# MCP2007
# Inequalities for Multivariate Normal Probabilities of Nonsymmetric Rectangles

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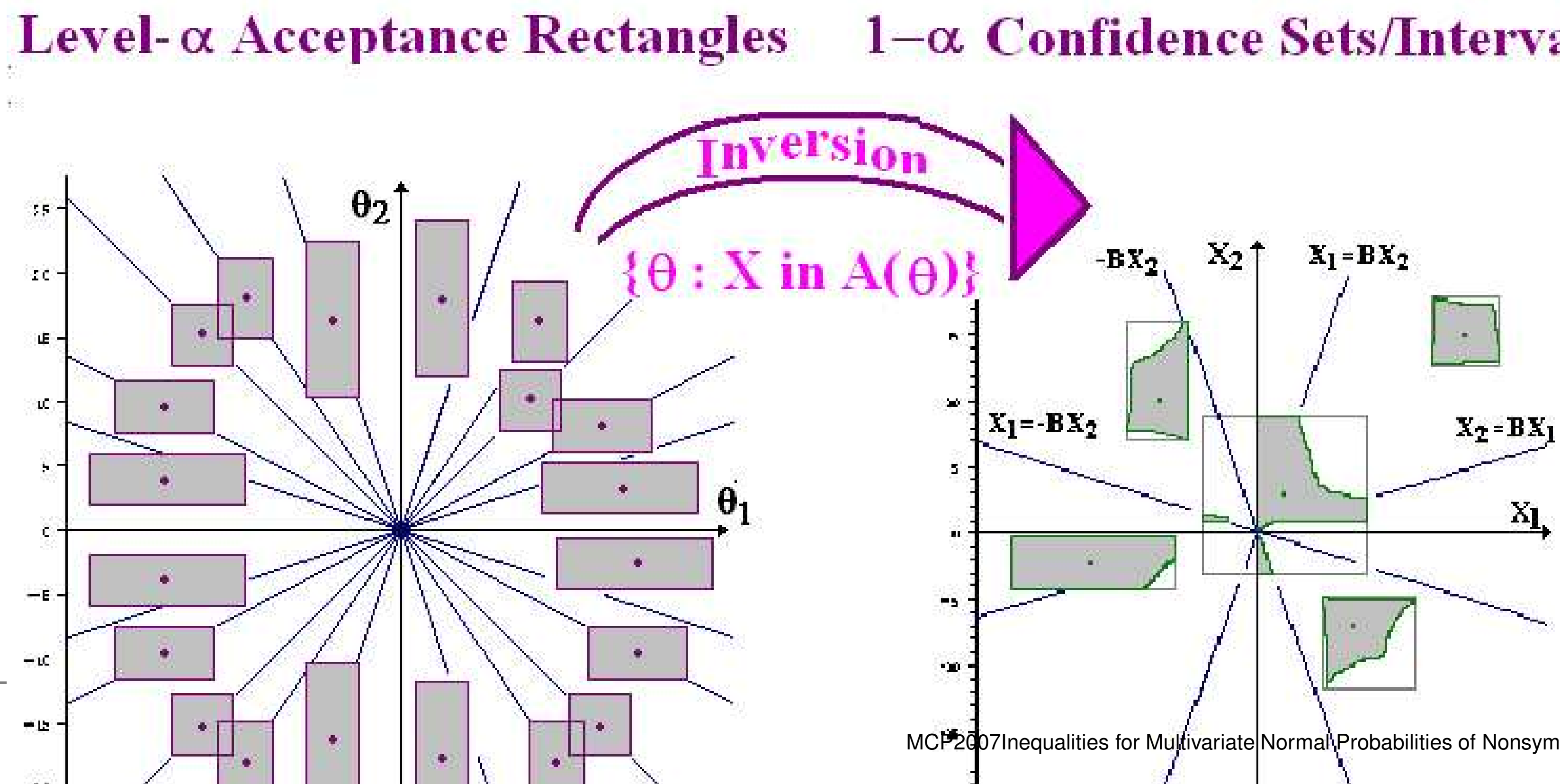
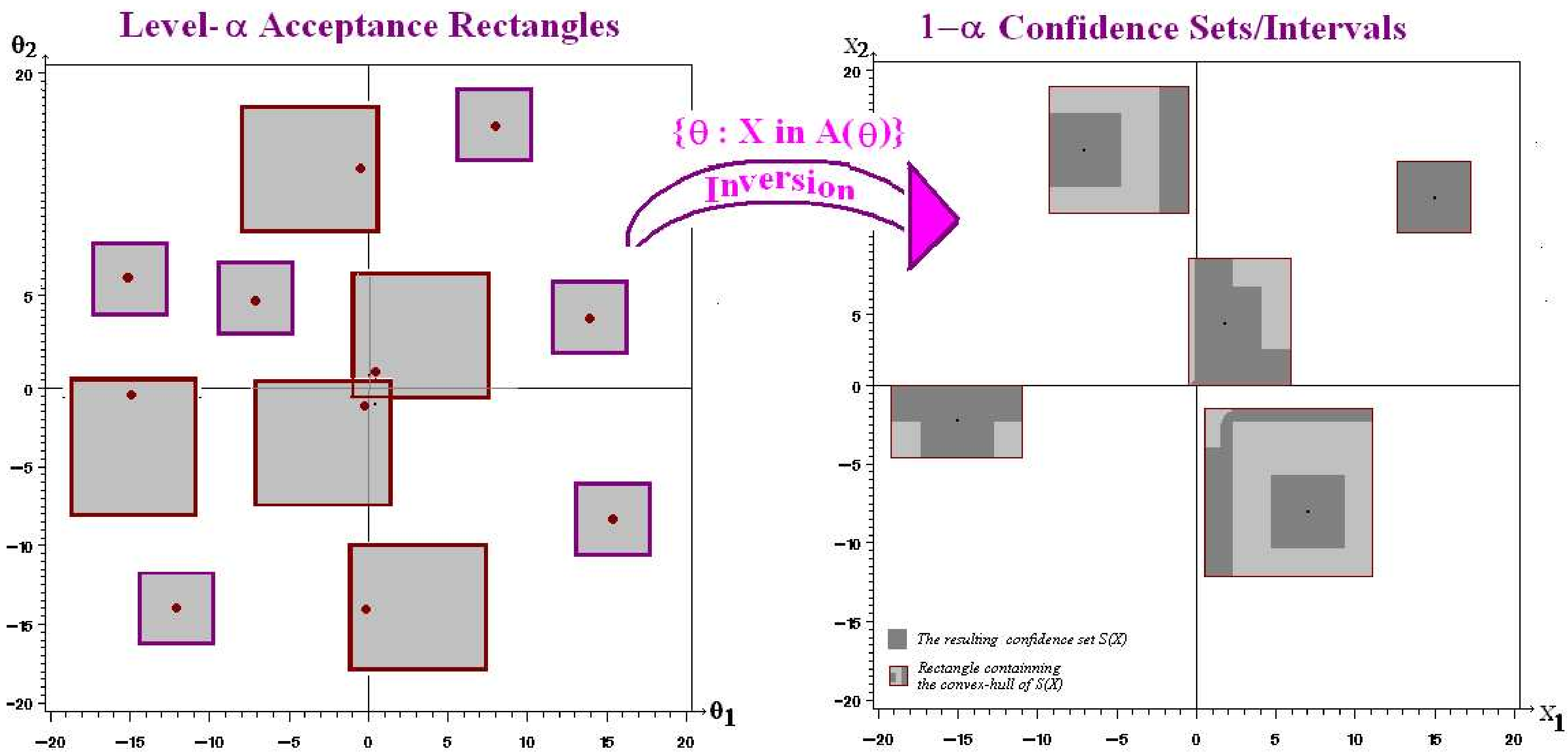
# The Variable-Ratio Intervals for Better Sign Determination

- Rectangular acceptance regions whose aspects-ratio is changing are inverted to form confidence set/intervals that are uniformly shorter than the conventional two-sided intervals for many situations.
    - The resulting intervals were uniformly shorter than those of Benjamini & Stark (1996). Still the resulting intervals could not compete with the conventional two-sided intervals when all $\{\mu_j\}_{j=1}^{n}$ where close to zero.



# The Quasi-Conventional intervals Intervals for Better Sign Determination

Rectangular acceptance regions with minimal incursion to the negative(positive) sides (and length constrains) inverted to form more powerful confidence intervals than the two-sided conventional intervals. Not necessarily symmetric rectangles!



# Are the Intervals Valid under Dependency?

When is a rectangular acceptance region conservative for $X_j \sim \mathcal{N}(\mu_j, \sigma_j)$? [a]

(1) For symmetric regions, Šidák's inequality (1967):

$$\Pr\left(\bigcap_{j=1}^{n}\left\{\frac{|X_j-\mu_j|}{\sigma_j} \le b_j\right\}\right) \ge \prod_{j=1}^{n}\Pr\left(\frac{|X_j-\mu_j|}{\sigma_j} \le b_j\right).$$

$\Rightarrow$ The Variable-Ratio intervals are conservative.

(2) For one asymmetric side (proof below),

$$\Pr\left(-a_1 \le \tfrac{X_1-\mu_1}{\sigma_1} \le b_1,\ \tfrac{|X_j-\mu_j|}{\sigma_j} \le b_j,\ j \ge 2\right)$$
$$\ge \Pr\left(-a_1 \le \tfrac{X_1-\mu_1}{\sigma_1} \le b_1\right)\prod_{j=2}^{n}\Pr\left(\frac{|X_j-\mu_j|}{\sigma_j} \le b_j\right).$$

(3) When $\Pr\left(\bigcap_{j=1}^{n}\left\{-a_j \le \frac{X_j-\mu_j}{\sigma_j} \le b_j\right\}\right) \ge \prod_{j=1}^{n}\Pr\left(-a_j \le \frac{X_j-\mu_j}{\sigma_j} \le b_j\right)$? In the case of the Quasi-Conventional intervals: $a_j + b_j \equiv C\ \forall j$.

[a] $a_j, b_j > 0\ \forall j$

# Outline of the Proof for the Asymmetric Inequality

For simplicity, assume that $\mu_j = 0$ and $\sigma_j = 1$, and denote
$P \equiv \Pr(-a_1 \leq X_1 \leq b_1, |X_j| \leq b_j,\ j \geq 2)$.
Using the symmetry of the region $\mathop{\times}\limits_{j=2}^{n} [-b_j, b_j]$ we can write:

$$P = \frac{1}{2}\Pr(|X_1| \leq b_1, |X_j| \leq b_j,\ j \geq 2) + \frac{1}{2}\Pr(|X_1| \leq a_1, |X_j| \leq b_j,\ j \geq 2).$$

Applying Šidák's inequality on both summands:

$$P \geq \left[\frac{1}{2}\Pr(|X_1| \leq b_1) + \frac{1}{2}\Pr(|X_1| \leq a_1)\right] \prod_{j=2}^{n} \Pr(|X_j| \leq b_j),$$

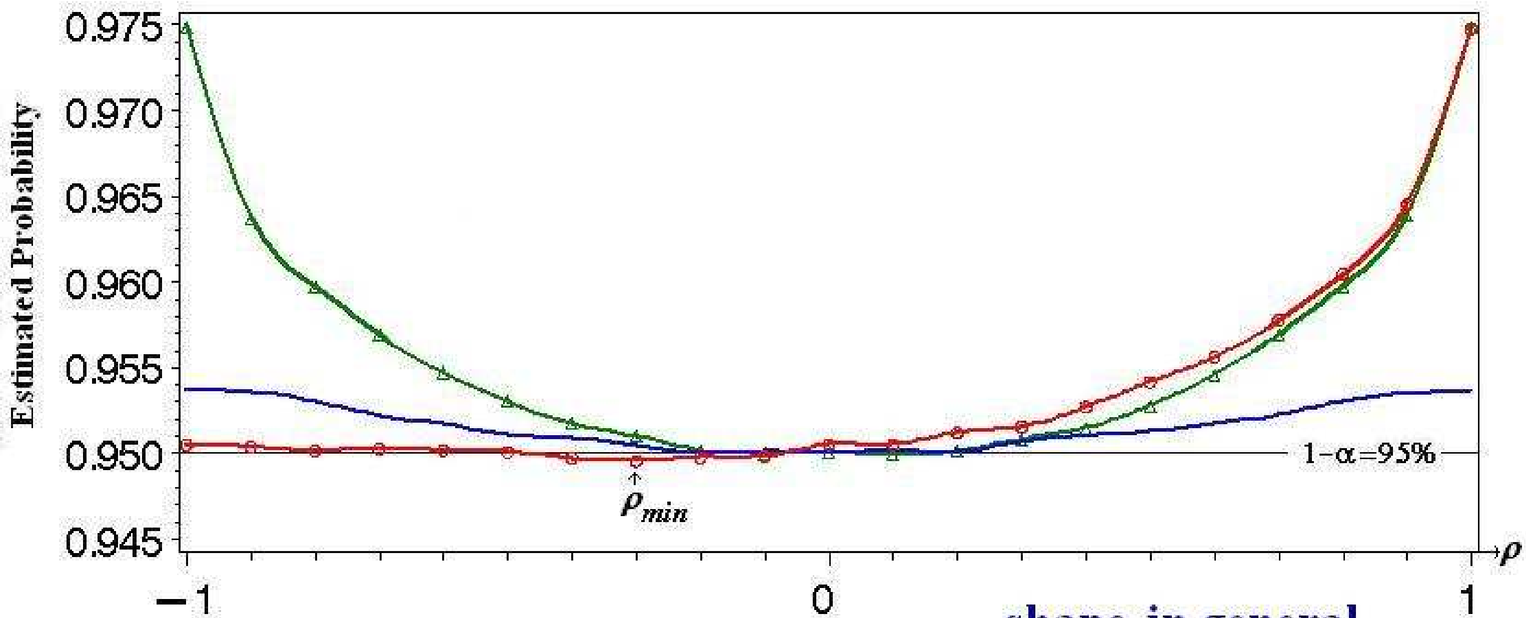
and use symmetry again,

$$\frac{1}{2}\Pr(|X_1| \leq b_1) + \frac{1}{2}\Pr(|X_1| \leq a_1) = \Pr(-a_1 \leq X_1 \leq b_1).$$

QED.

# Monte-Carlo estimation of the probability to be inside a Quasi-Conventional acceptance square ($r=1.2$, $1-\alpha=95\%$)

for dependent $X_1, X_2 \sim N(0,1)$ with correlation $\rho$

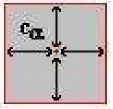


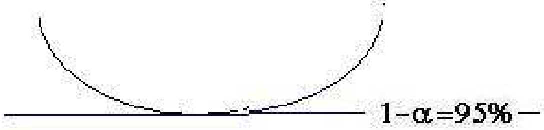

**shape in general**

**Typical Quasi-Conventional acceptace regions**

The Conventional symmetric square

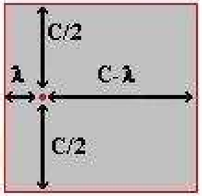
$\hat{P}(|X_1| \leqslant c_\alpha, |X_2| \leqslant c_\alpha) \geqslant 1-\alpha$

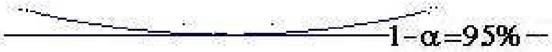



Square with one asymmetric side

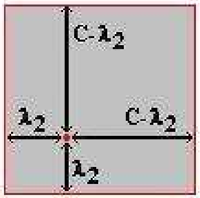
$\hat{P}(-\lambda \leqslant X_1 \leqslant C-\lambda, |X_2| \leqslant C/2) \geqslant 1-\alpha$

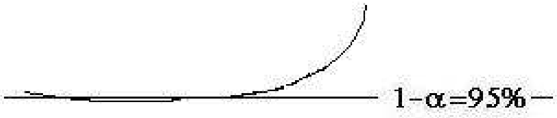



Square with two asymmetric sides

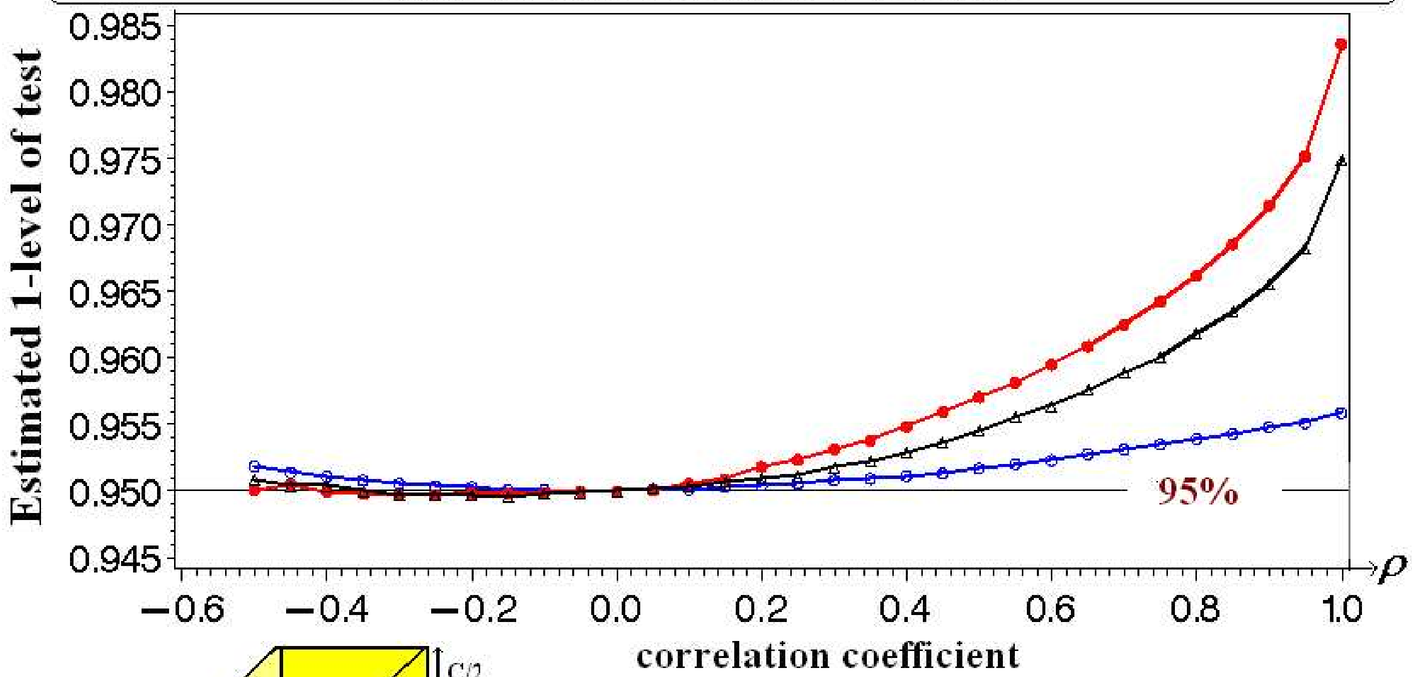
$\hat{P}(-\lambda_2 \leqslant X_1 \leqslant C-\lambda_2, -\lambda_2 \leqslant X_2 \leqslant C-\lambda_2)$



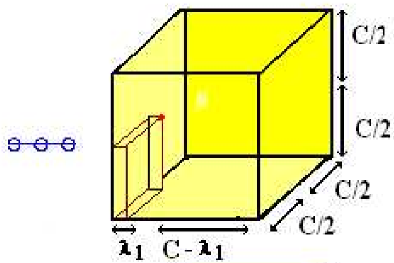



# Monte-Carlo Estimation of 1-level of test perfored by asymmetric cubical acceptance regions

**3 equicorrelated standard normals**

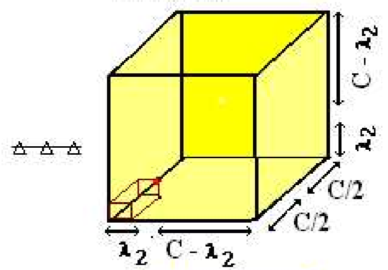



**Acceptance Regions**




$\hat{P}(-\lambda_1 \leq X_1 \leq C-\lambda_1, |X_2|, |X_3| \leq C/2) \geq 0.9500$

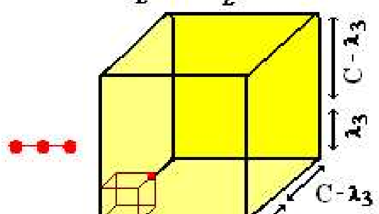
$\lambda_1 = 1.754$




$\hat{P}(-\lambda_2 \leq X_1, X_2 \leq C-\lambda_2, |X_3| \leq C/2) \geq 0.9496$

SD = 0.0005

$\lambda_2 = 1.995$




$\hat{P}(-\lambda_3 \leq X_1, X_2, X_3 \leq C-\lambda_3) \geq 0.9497$

$\lambda_3 = 2.119 \quad C=5.73$

For level-5% acceptance cubes (under independence)

# The Bivariate Case

For $X_1, X_2 \sim \mathcal{N}(0,1)$ with $\rho = Corr(X_1, X_2)$, if $a_j \leq b_j$, $j = 1, 2$.

1. For $\rho \geq 0$,

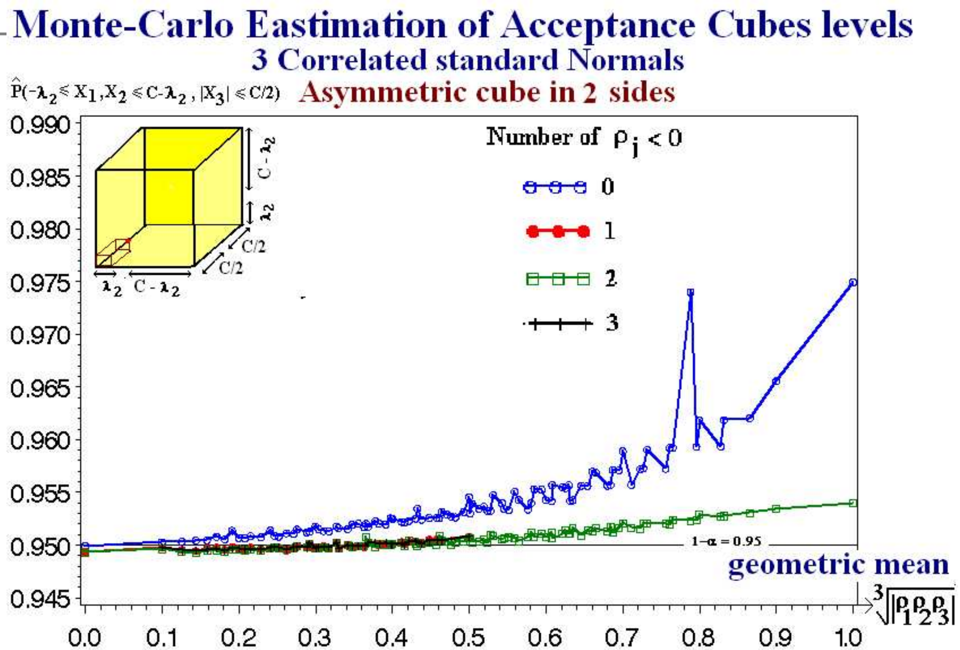
$$\begin{aligned} &\Pr\left(-a_1 \leq X_1 \leq b_1,\, -a_2 \leq X_2 \leq b_2\right) \\ &\geq \Pr\left(-a_1 \leq X_1 \leq b_1\right) \cdot \Pr\left(-a_2 \leq X_2 \leq b_2\right). \end{aligned}$$

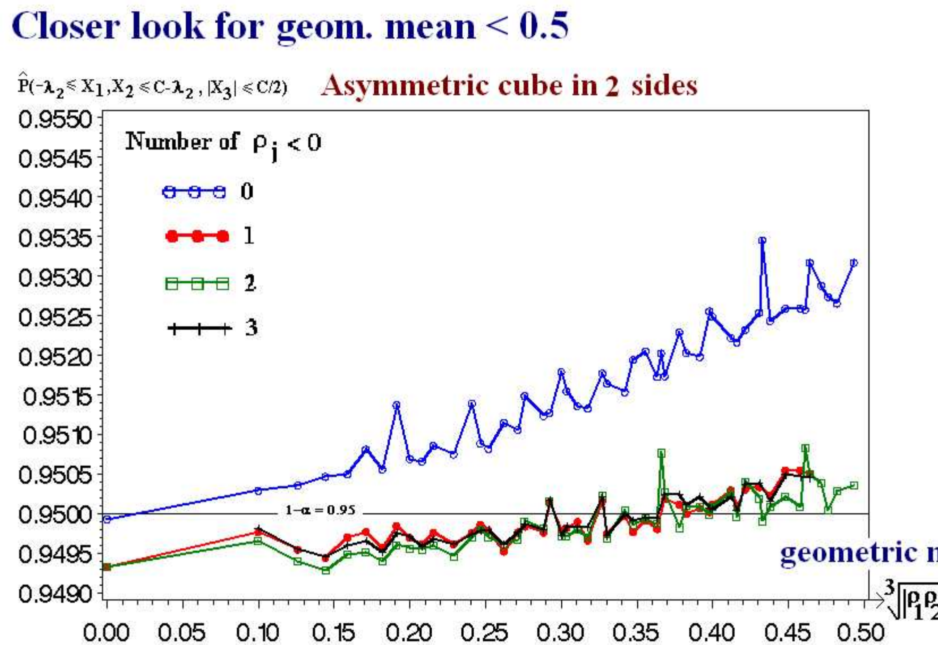
2. $\Pr\left(-a_1 \leq X_1 \leq b_1,\, -a_2 \leq X_2 \leq b_2\right)$ gets its minima for $\rho \in \left[\max\left\{-\frac{b_1-a_1}{2a_2}, -\frac{b_2-a_2}{2a_1}, -1\right\}, 0\right)$. This lower end is very close to the true value $\rho_{min}$.

For general $n$, under positive common correlation coefficient we have exchangeable r.v. Hence, by Tong (1977), if $Cov(X_i, X_j) = \rho$, $\forall i < j$, then

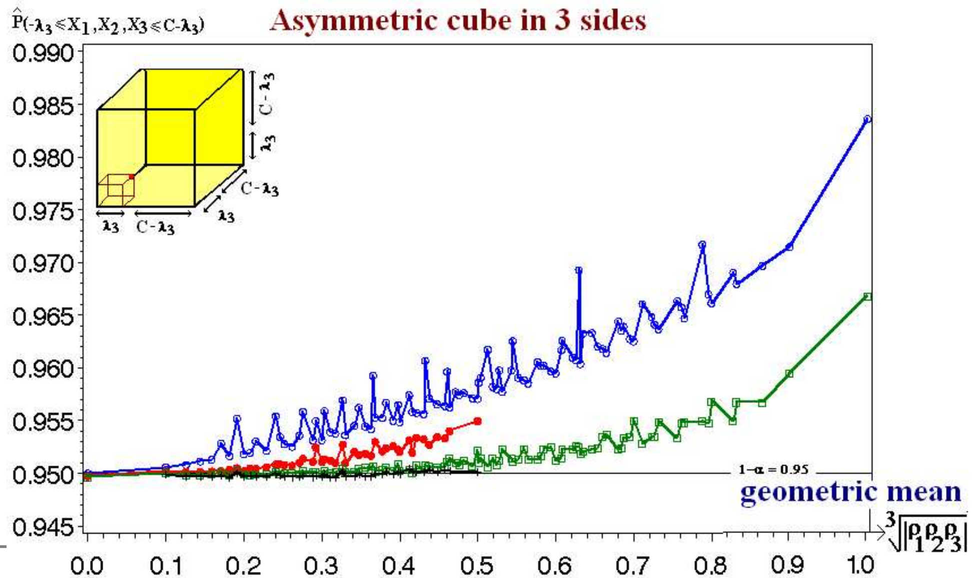
$$\Pr\left(\bigcap_{j=1}^{n}\left\{-a_j \leq \frac{X_j - \mu_j}{\sigma_j} \leq b_j\right\}\right) \geq \prod_{j=1}^{n} \Pr\left(-a_j \leq \frac{X_j - \mu_j}{\sigma_j} \leq b_j\right).$$

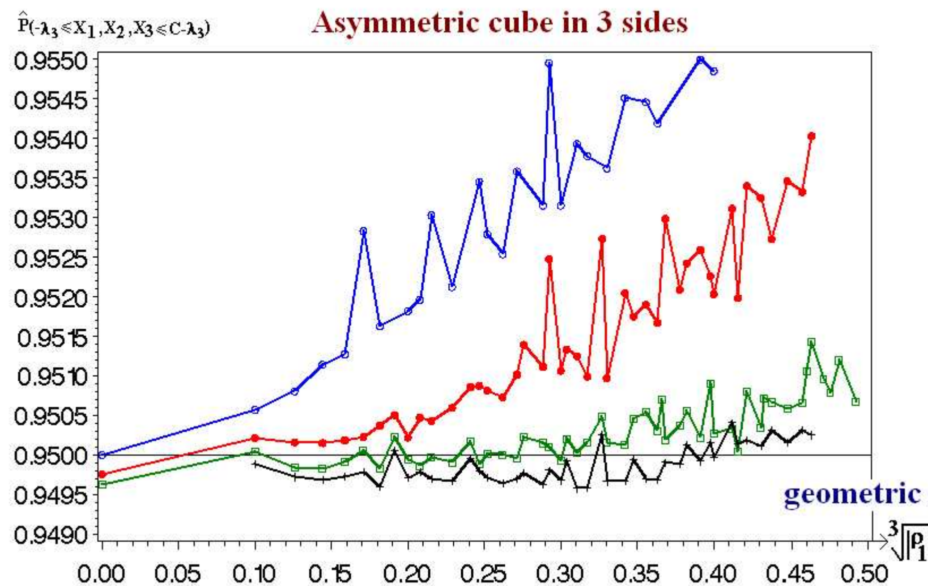
# n=3 - Estimated $1-$level. vs the geom. mean of $|\rho_{ij}|$




min $\hat{P}(-\lambda_2 \leq X_1, X_2 \leq C-\lambda_2, |X_3| \leq C/2) = 0.94929(\pm 0.0005)$ obtained for $\rho_1=-0.3, \rho_2=0.1, \rho_3=-0.1$ Geo. mean= 0.144




min $\hat{P}(-\lambda_3 \leq X_1, X_2, X_3 \leq C-\lambda_3) = 0.94958(\pm 0.0005)$ obtained for $\rho_1=-0.3, \rho_2=-0.2\ \rho_3=-0.5$ Geo. mean= 0.311

# An Extension to Elliptically Contoured Distributions

Das Gupta *et al* (1972) extended Šidák's inequality for elliptically contoured distributions(studentized $t$, logistic, ...):

**Theorem 1 (Das Gupta *et al* (1972))** *Suppose that* $(Z_1, Z_2, \cdots, Z_n)$ *have multivariate elliptically contoured distribution and an nonsingular correlation matrix* $\mathbf{R} = \{\rho_{ij}\}$*. If* $\mathbf{C}$ *is a symmetric set in* $\Re^{n-1}$*, then for any* $b > 0$*,*

$$\Pr\left(|Z_1| \leq b,\, (Z_2, \cdots, Z_n) \in \mathbf{C}\right) \geq \Pr\left(|Z_1| \leq b\right) \cdot \Pr\left((Z_2, \cdots, Z_n) \in \mathbf{C}\right).$$

Considering the symmetry of the joint probability inside $\mathbf{C}$:

**Theorem 2** *Under the assumptions of Theorem 1. For any* $a, b > 0$*,*

$$\Pr\left(-a \leq Z_1 \leq b,\, (Z_2, \cdots, Z_n) \in \mathbf{C}\right) \geq \Pr\left(-a \leq Z_1 \leq b\right) \cdot \Pr\left((Z_2, \cdots, Z_n) \in \mathbf{C}\right)$$

**Result 1** *Letting* $a_1 \to \infty$ *we obtain:*

$$\Pr\left(Z_1 \leq b_1,\, (Z_2, \cdots, Z_n) \in \mathbf{C}\right) \geq \Pr\left(Z_1 \leq b_1\right) \cdot \Pr\left((Z_2, \cdots, Z_n) \in \mathbf{C}\right).$$

All inequalities also hold for singular correlation matrix (Šidák, 1967).

# Results for General Asymmetric Rectangles

Monte-Carlo study (for $0 < a_j \leq b_j = C - a_j$) show:

(1) For positively associated normals ($\rho_{ij} \geq 0, \forall i < j$)

$$\Pr\left(\bigcap_{j=1}^{n}\left\{-a_j \leq \frac{X_j - \mu_j}{\sigma_j} \leq b_j\right\}\right) \geq \prod_{j=1}^{n} \Pr\left(-a_j \leq \frac{X_j - \mu_j}{\sigma_j} \leq b_j\right) = 1-c$$

(2) For non-positively associated normals - negligible liberalization.

# Reference

Benjamini, Y., Hochberg, Y., and Stark, P.B. (1998). Confidence intervals with more power to determine the sign: two ends constrain the means. *J. Amer. Statist. Assoc.* **93**, 309-317.

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Tong, Y.L. (1977). An ordering theorem for conditionally independent and identically distributed random variables. *Ann. Statist.* **5**, 274-277.