An Application of the Closed Testing Principle to Enhance One-Sided Confidence Regions for a Multivariate Location Parameter

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Outline

Introduction

Direct Derivation of Confidence Regions for $\theta$

Enhanced Confidence Regions for $\theta$
One-Sided Confidence Regions

- $X_1, \ldots, X_n$ i.i.d. random vectors in $\mathbb{R}^p$
- $X_i \sim P_\vartheta$, $\vartheta \in \Theta$ unknown
- $P_\vartheta$ (at least directionally) symmetric w.r.t. $\vartheta$

Problem: Find a $1 - \alpha$ confidence region for $\vartheta$ that is

- as strict as possible in specific directions
- possibly unbounded in “irrelevant” directions

(e.g. a cone or an orthant).
Connection with One-Sided Location Tests

Let $\varphi_{\alpha}$ be a non-randomized level $\alpha$ test for

$$H_0 : \vartheta \in \Theta_0(\gamma) \quad \text{vs.} \quad H_1 : \vartheta \in \Theta \setminus \Theta_0(\gamma).$$

(E. g. $\Theta_0(\gamma) = \gamma + (-\infty, 0]^p$)

Inversion of $\varphi_{\alpha}$

$$\Rightarrow C_{1-\alpha}(X_1, \ldots, X_n) = \{\gamma : \varphi_{\alpha}((X_1, \ldots, X_n), \gamma) = 0\}, \text{ and}$$

$$P_{\vartheta}(C_{1-\alpha}(X) \ni \gamma) \geq 1 - \alpha \quad \forall \vartheta \in \Theta_0(\gamma) \quad \forall \gamma \in \Theta.$$

$C_{1-\alpha}$ is a $1 - \alpha$ confidence region for the meta-parameter $\gamma$. 

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Direct Derivation of Confidence Regions for $\vartheta$

Assume that $\gamma \in \Theta_0(\gamma), \forall \gamma \in \Theta$.
Then

$$P_{\vartheta}(C_{1-\alpha}(X) \ni \vartheta) \geq 1 - \alpha \quad \forall \vartheta \in \Theta.$$ 

$C_{1-\alpha}$ is also a $1 - \alpha$ confidence region for the location parameter $\vartheta$. 

Problems:
• conservative
• unpleasant shape – to be illustrated.
Direct Derivation of Confidence Regions for $\theta$

Assume that $\gamma \in \Theta_0(\gamma), \forall \gamma \in \Theta$.

Then

$$P_\theta(C_{1-\alpha}(X) \ni \theta \geq 1 - \alpha \quad \forall \theta \in \Theta.$$ 

$C_{1-\alpha}$ is also a $1 - \alpha$ confidence region for the location parameter $\theta$.

Problems:

- conservative
- unpleasant shape – to be illustrated...
Min and Max Tests

**Min Test**
Reject $H_0 : \exists j \in \{1, \ldots, p\} : \vartheta_j \leq \gamma_j$ in favor of $H_1 : \vartheta > \gamma$ at the level $\alpha$ if and only if

$$\varphi_{j,\alpha}((X_{1j}, \ldots, X_{nj}), \gamma_j) = 1, \forall j \in 1, \ldots, p.$$ 

**Bonferroni Max Test**
Reject $H_0 : \vartheta \leq \gamma$ in favor of $H_1 : \exists j \in \{1, \ldots, p\} : \vartheta_j > \gamma_j$ at the level $\alpha$ if and only if

$$\exists j \in 1, \ldots, p : \varphi_{j,\alpha/p}((X_{1j}, \ldots, X_{nj}), \gamma_j) = 1.$$
Example

Two variables of the pulmonary function data by Randles (1989) (slightly modified from Merchant et al., 1975).
Pulmonary Function Data, Wilcoxon Min Test

\[ C_{1-\alpha}(X) = c_{1-\alpha}(X) - \Theta_0(0) \]
P. F. Data, Wilcoxon Bonferroni Max Test

\[ C_{1-\alpha}(X) = c_{1-\alpha}(X) - \Theta_0(0) \]
P. F. Data, Sign Test by Larocque/Labarre (2004)

\[ C_{1-\alpha}(X) \approx c_{1-\alpha}(X) - \Theta_0(0) \text{ (outside a sufficiently large ball)} \]
Back to the Drawbacks of the Direct Approach

If confidence regions for $\gamma$ are directly used as confidence regions for $\theta$, they are

- usually conservative and
- similar in shape to $-\Theta_0$, rather than to $\Theta_1 = \Theta \setminus \Theta_0$. 
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Enhanced Confidence Regions for $\vartheta$ (1)

Temptation: $\tilde{C}_{1-\alpha}(X) = \bigcap_{\gamma \notin C_{1-\alpha}(X)} \Theta_1(\gamma)$
Enhanced Confidence Regions for $\theta$ (1)

Temptation: $\tilde{C}_{1-\alpha}(X) = \bigcap_{\gamma \notin C_{1-\alpha}(X)} \Theta_1(\gamma)$

$\Rightarrow$ liberal – multiple testing problem!

Solution: Reduce the set of possible meta-parameters in advance.
Enhanced Confidence Regions for $\vartheta$ (2)

Let $C_{1-\alpha} : X \rightarrow \mathcal{P}(\mathbb{R}^p)$ be a $1 - \alpha$ confidence region for $\gamma$ based on $(\Theta_0(\gamma))_{\gamma \in \mathbb{R}^p}$.
Let $\Theta_0(\gamma) = \gamma + \Theta_0(0), \forall \gamma \in \mathbb{R}^p$, closed, $\Theta_1(\gamma) = \mathbb{R}^p \setminus \Theta_0(\gamma)$.
Assume that $\Theta_0(\gamma) \subset \Theta_0(\gamma + (\delta, \ldots, \delta)^T), \forall \gamma \in \mathbb{R}^p, \delta > 0$.
With $\gamma_i = (i, \ldots, i)^T \in \mathbb{R}^p, \forall i \in I = [\ell, \infty)$, define
$$\tilde{C}_{1-\alpha}(X) := \bigcap_{i \in I : \gamma_i \notin C_{1-\alpha}(X) \forall i' \leq i} \Theta_1(\gamma_i).$$

Then
$$P_\vartheta \left( \tilde{C}_{1-\alpha}(X) \ni \vartheta \right) \geq 1 - \alpha \quad \forall \vartheta \in \mathbb{R}^p.$$
Idea of the Proof

- $(\Theta_0(\gamma_i))_{i \in I}$ is closed under (finite and infinite) intersections.
- Apply the closed testing principle (Marcus, Peritz, and Gabriel, 1976).
- Translate to confidence regions.
Pulmonary Function Data, Wilcoxon Min Test

\[ C_{1-\alpha}(X) \]
**Pulmonary Function Data, Wilcoxon Min Test**

![Diagram of pulmonary function data with Wilcoxon Min Test](image)

Introduction

Direct Confidence Regions for $\vartheta$

Enhanced Confidence Regions for $\vartheta$
Pulmonary Function Data, Wilcoxon Min Test

\[
\begin{align*}
C_{1-\alpha}(X) & \quad \tilde{C}_{1-\alpha}(X)
\end{align*}
\]
Enhanced Confidence Regions: Properties

- \( \tilde{C}_{1-\alpha}(X) = c_{1-\alpha}(X) + \Theta_1(0) \) (by definition)
- \( \tilde{C}_{1-\alpha}(X) \subset C_{1-\alpha}(X) \) under suitable conditions (\( \Theta_1(0) \) convex cone, translation invariance, and a monotonicity property)
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  (\( \Theta_1(0) \) convex cone, translation invariance, and a monotonicity property)

Disadvantage: Restricted set of possible confidence regions (search on the diagonal)
Summary

- A confidence region $C_{1-\alpha}(X)$ obtained by inversion of a test for a composite null hypothesis is for a meta-parameter.
- Even if $C_{1-\alpha}(X)$ may also be a confidence region for the parameter $\vartheta$ itself, it is not very useful.
- The proposed method based on the closed testing principle yields a confidence region $\tilde{C}_{1-\alpha}(X)$ with a more useful shape.
- $\tilde{C}_{1-\alpha}(X)$ is also less conservative than $C_{1-\alpha}(X)$ under suitable conditions.
References


