New adaptive procedures that control the FDR

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- $(\mathcal{X}, \mathfrak{X}, \mathbf{P})$ probability space.
- $\mathcal{H}$ a finite set of hypotheses for $\mathbf{P}$, $m := |\mathcal{H}|$ (known)

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- For each $h \in \mathcal{H}$, p-value: $p_h : \mathcal{X} \to [0,1]$ (measurable) such that

$$\text{If } h \in \mathcal{H}_0,\ \ \forall t \in [0,1], \mathbf{P}(p_h \le t) \le t$$

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- A multiple testing procedure: a (measurable) function

$$R : \mathbf{p} = (p_h)_{h \in \mathcal{H}} \in [0,1]^{\mathcal{H}} \mapsto R(\mathbf{p}) \subset \mathcal{H}$$

(returns the rejected hypotheses)

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How to build R with $\mathrm{FDR}(R) \leq \alpha$?

Step-up procedure

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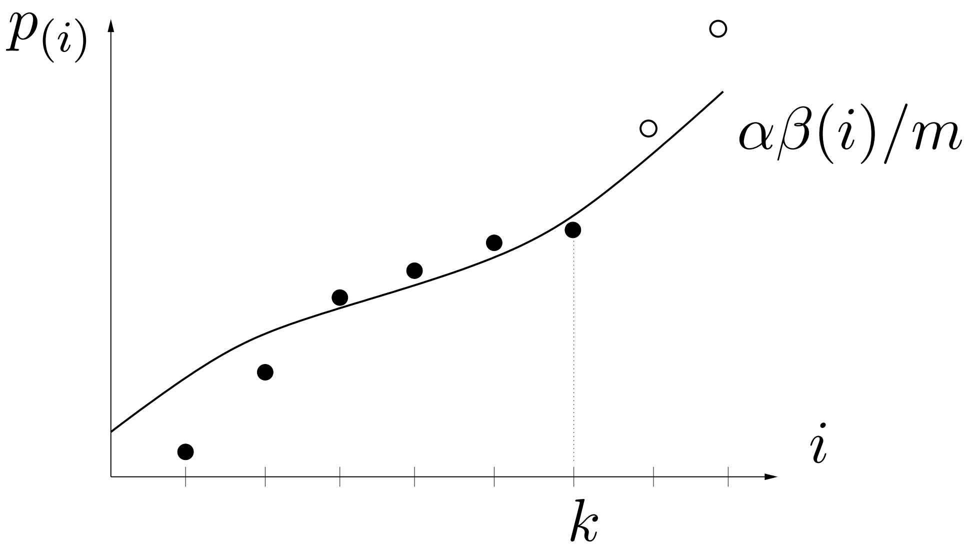

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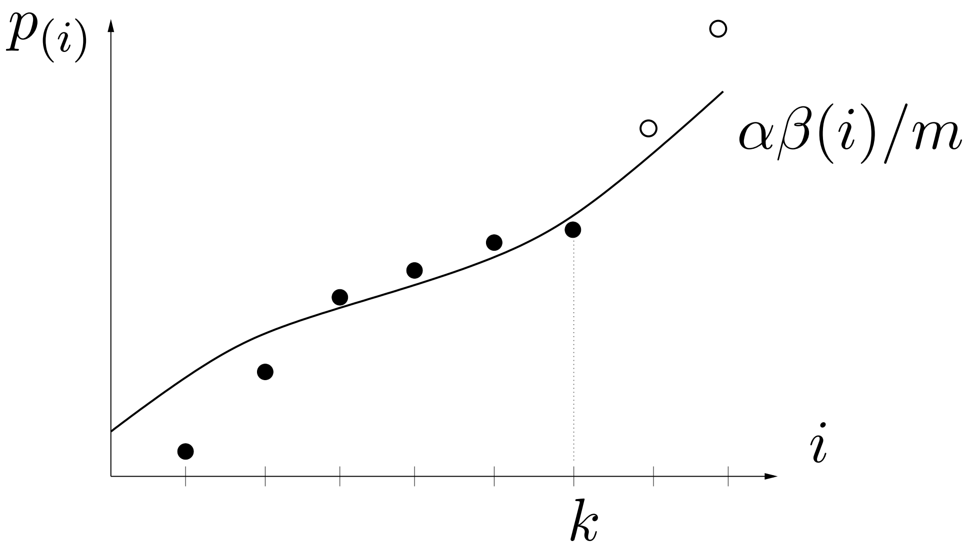

Definition (step-up procedure with threshold function β)

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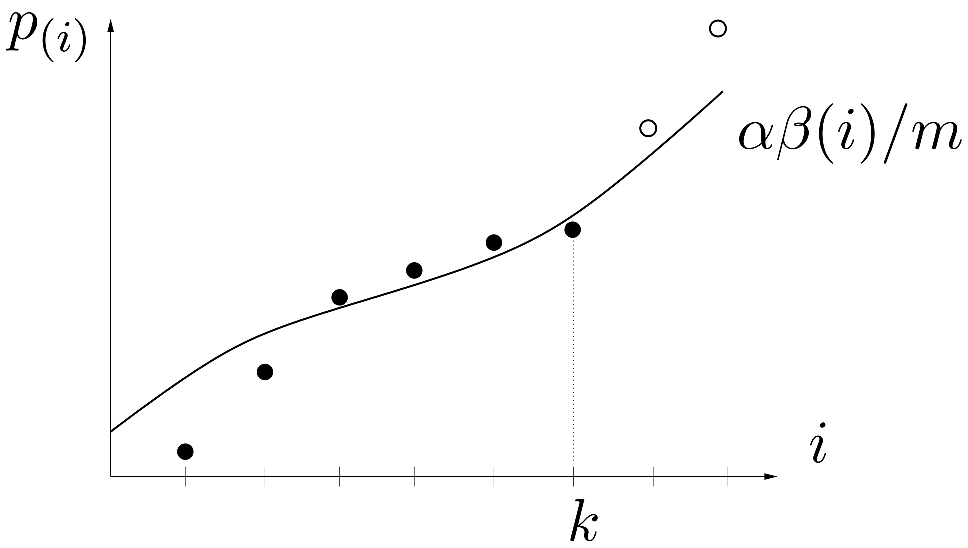

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Linear step-up procedure: R_β where $\beta(i) = i$.

Goal in step-up FDR control

Find a threshold function β such that

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With β as "large" as possible

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Theorem 1 p-values independent or positively dependent (PRDS):
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Theorem 2 p-values with general dependencies:
The step-up procedure with threshold function $\beta(i) = i/(1 + 1/2 + \cdots + 1/m)$ satisfies

$$\mathrm{FDR}(R_\beta) \leq \pi_0\alpha.$$

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If β of the form:

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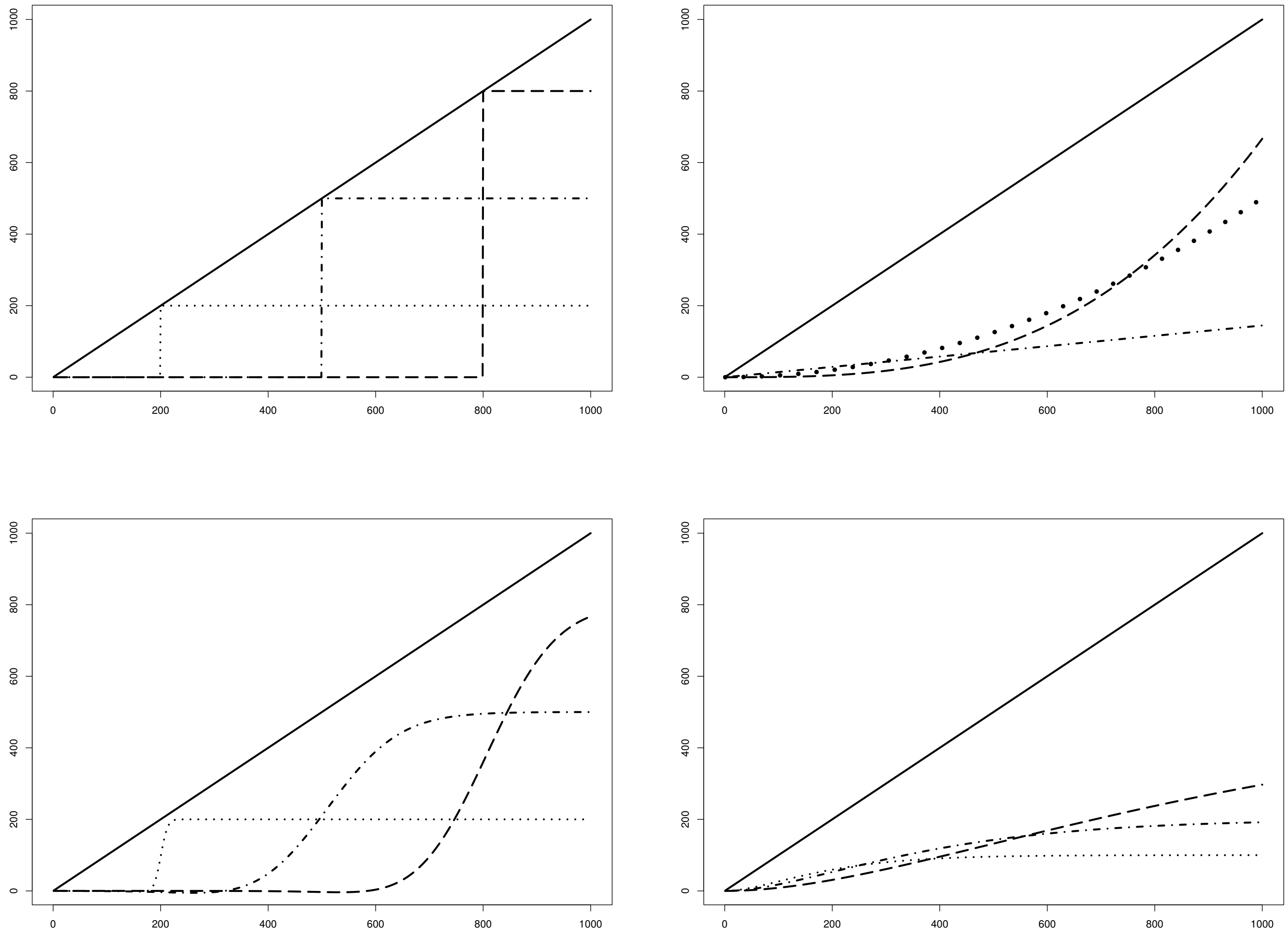
- $\nu(\{k\}) \propto k$ gives $\beta(i) = \frac{i(i+1)(2i+1)}{3m(m+1)}$

Some choices for β (general dependencies)



$\Rightarrow$ no uniformly-optimal choice (it depends on the data)

To control FDR under gen. dep.: not only BY's procedure !

Introduction to adaptive procedures

Summary: we have $\text{FDR}(R_\beta) \leq \pi_0 \alpha$ when

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(π_0)-Adaptive step-up procedures: $R_{\hat{\beta}}$ with $\hat{\beta} \simeq \beta^\star$

Our Goal

Find $\hat{\beta} \geq \beta$ such that $\mathrm{FDR}(R_{\hat{\beta}}) \leq \alpha$ and $\hat{\beta}$ "close to" $\beta\pi_0^{-1}$

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One-stage procedure: $\hat{\beta}$ is deterministic

Outline

I. When the p-values are independent
- Some existing adaptive procedures
- New adaptive procedures

II. When the p-values are dependent
- New (and first ?) adaptive procedures

I. Existing adaptive procedures with FDR control

[Benjamini, Krieger and Yekutieli (2006)] BKY06:

1. Apply the standard step-up linear procedure R_0 at level $\alpha/(1+\alpha)$ and put $\hat{F} = \frac{m}{m-|R_0|}$
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Using [Storey (2001)] Storey-λ :

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Benjamini, Krieger and Yekutieli (2006):

Theorem 4 p-values independent
These two procedures satisfy $\mathrm{FDR}(R) \le \alpha$

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The step-up procedure with global threshold $\frac{\alpha}{1+\alpha}\min\left(\frac{i}{m-i+1}, 1\right)$ has a FDR smaller than α.

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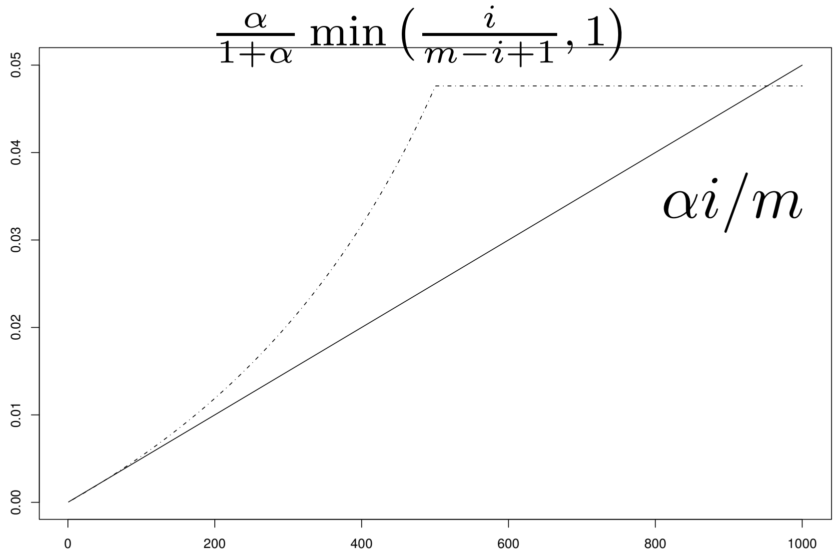

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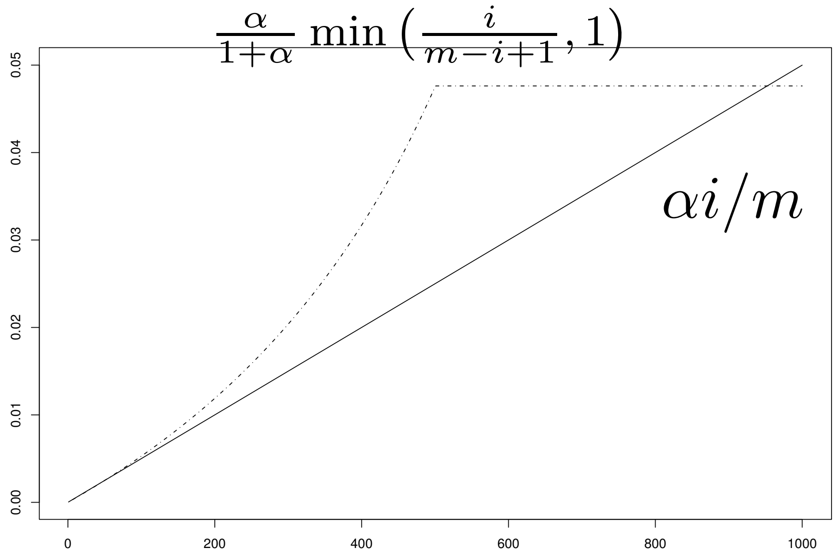

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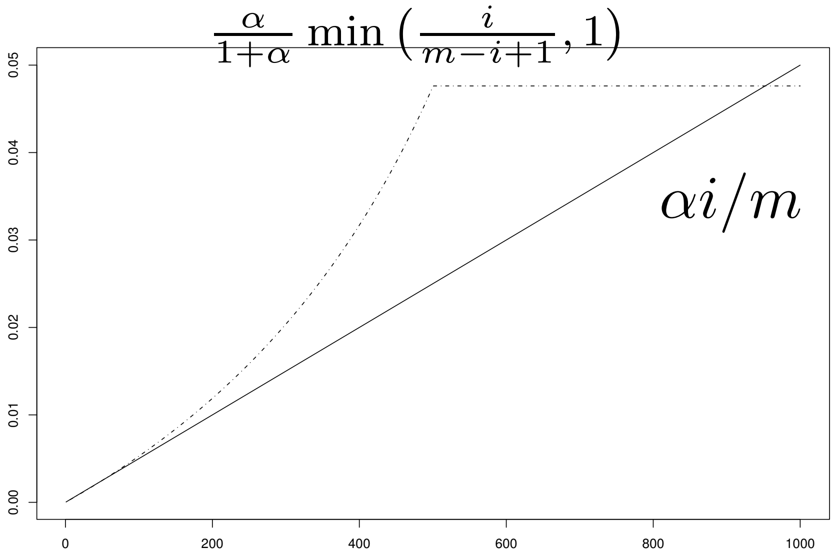

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Idea: use this procedure in the first step !

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Theorem 6 p-values independent: consider the two-stage procedure :

1. Apply the new one-stage adaptive procedure R_0' at level α and put $\hat{F} = \frac{m}{m-|R_0'|+1}$
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On simulations:
- independent case: Power?
Storey-$1/2 \gg$ BR07-2S $\gg$ BKY06
- positively dependent case: FDR control?
Storey-$1/2$ is not robust!
New two-stage procedure seems robust to positive correlations

I I. Under dependence

Recall: FDR control for step-up procedures with $\alpha\beta(i)/m$ in the cases:
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Loss/ indep case: $\frac{\alpha}{2}$, $\frac{\alpha}{4}$ and $\frac{1}{1-\sqrt{(2x-1)_+}} \leq \frac{1}{1-x}$

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$\Rightarrow$ interest more theoretical than practical.

Conclusion and future works

New adaptive procedures that control the FDR:

- when the p-values are independent:
  one-stage explicit and better than LSU
  two-stage $\longrightarrow$ better than BKY06
  $\longrightarrow$ seems robust to positive correlations.
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Choose the prior ν from an estimator of π_0?

Thank you for your attention!

A preprint is available on: http://genome.jouy.inra.fr/$\sim$eroquain

Appendix

I. Simulations

For $k = 1, \ldots, m$,

$$Y_k \sim \mathcal{N}(\mu_k, 1)$$

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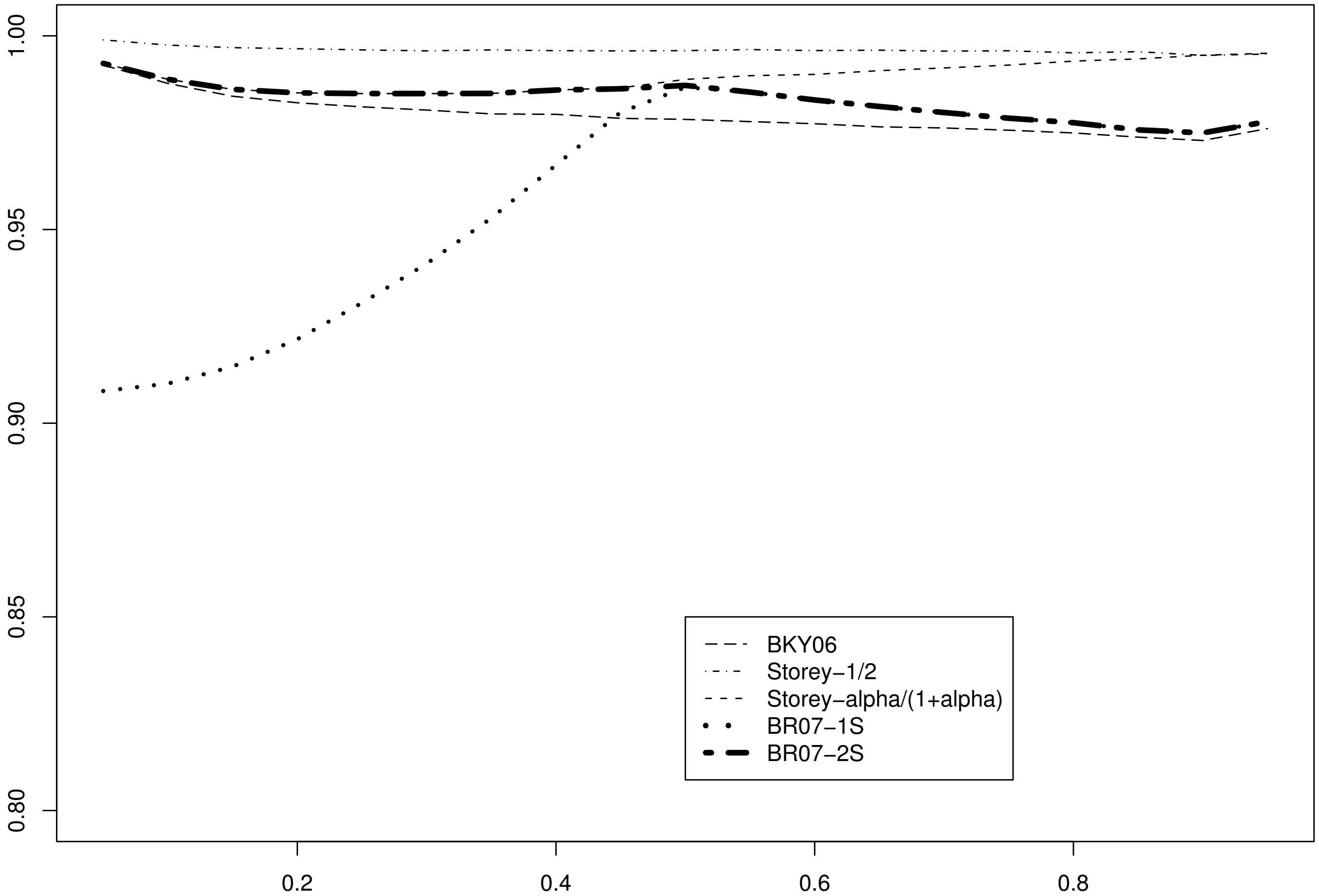
With 10000 simulations, $m = 100$, mean $= 3$:

- Power (in independent case) :
nb of correct rejections / nb of correct rejections of the oracle procedure
- FDR estimation (in the positive dependent case)

I. Simulations, Power, indep. case

$\rho = 0$ (independent case) :



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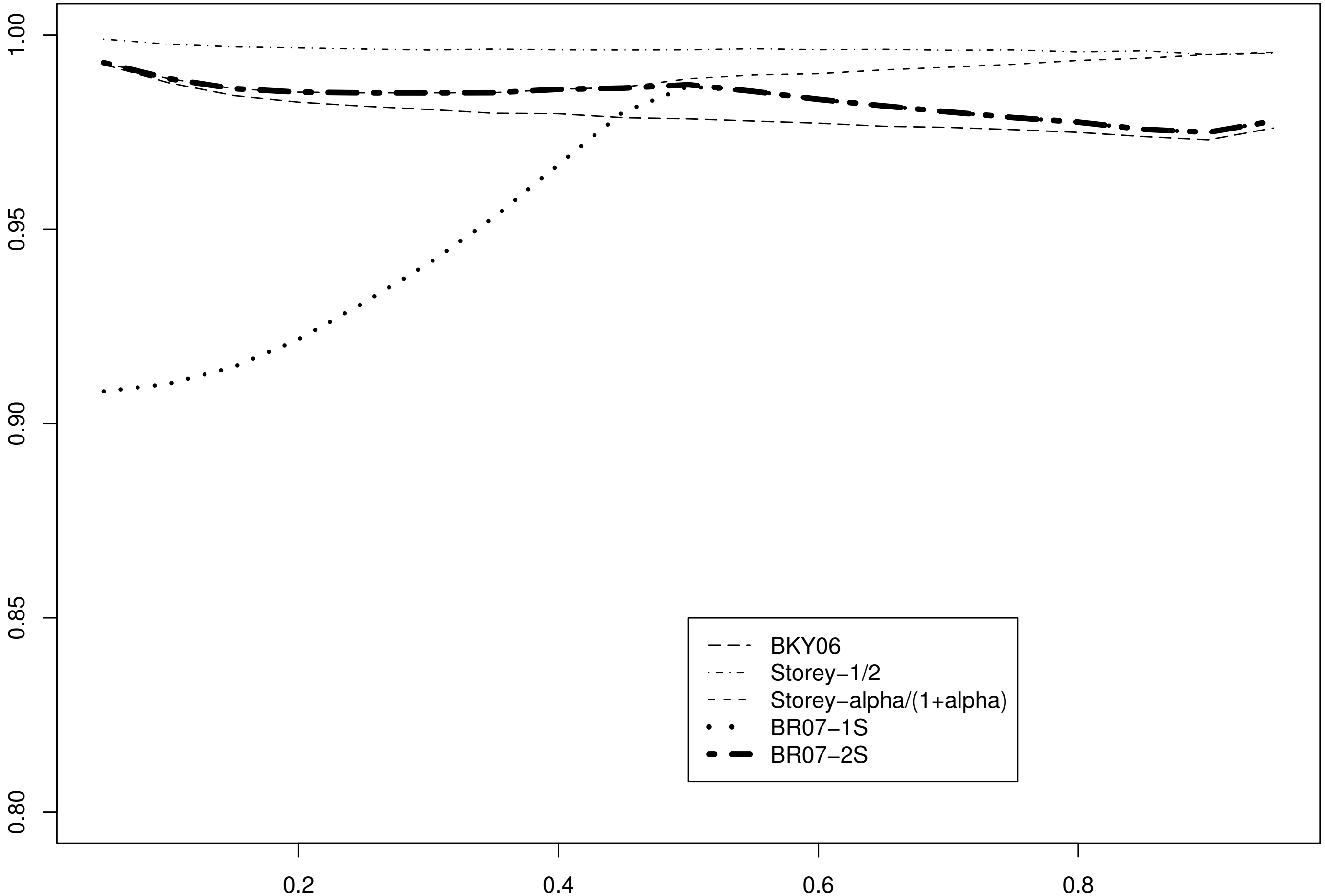
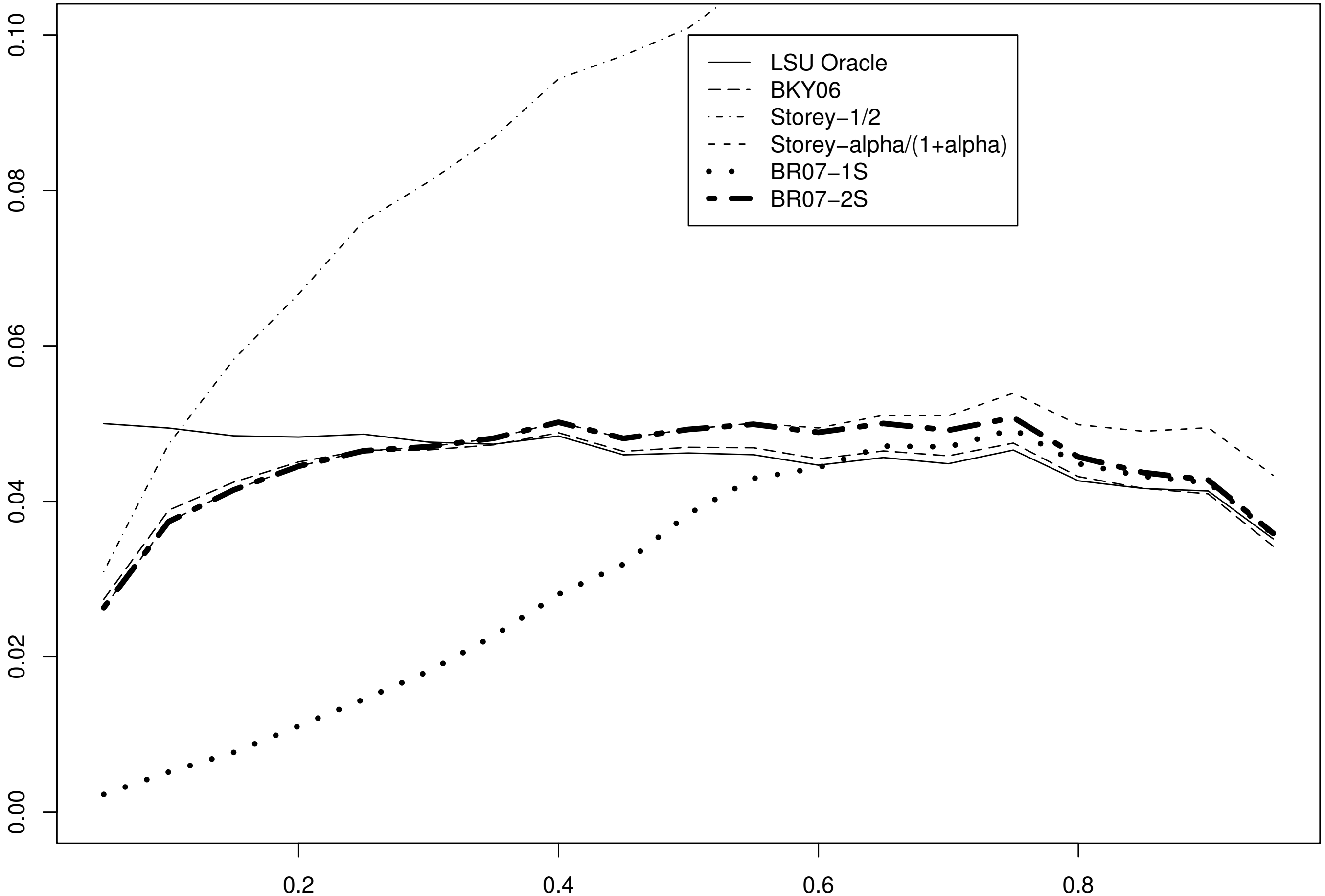


$$\text{Storey-}1/2 \gg \text{Storey-}\alpha/(1+\alpha) \gg \text{BR07-2S} \gg \text{BKY06}$$

I. Simulations, FDR, with corr

$\rho = 0.5$ (positive correlate case) :



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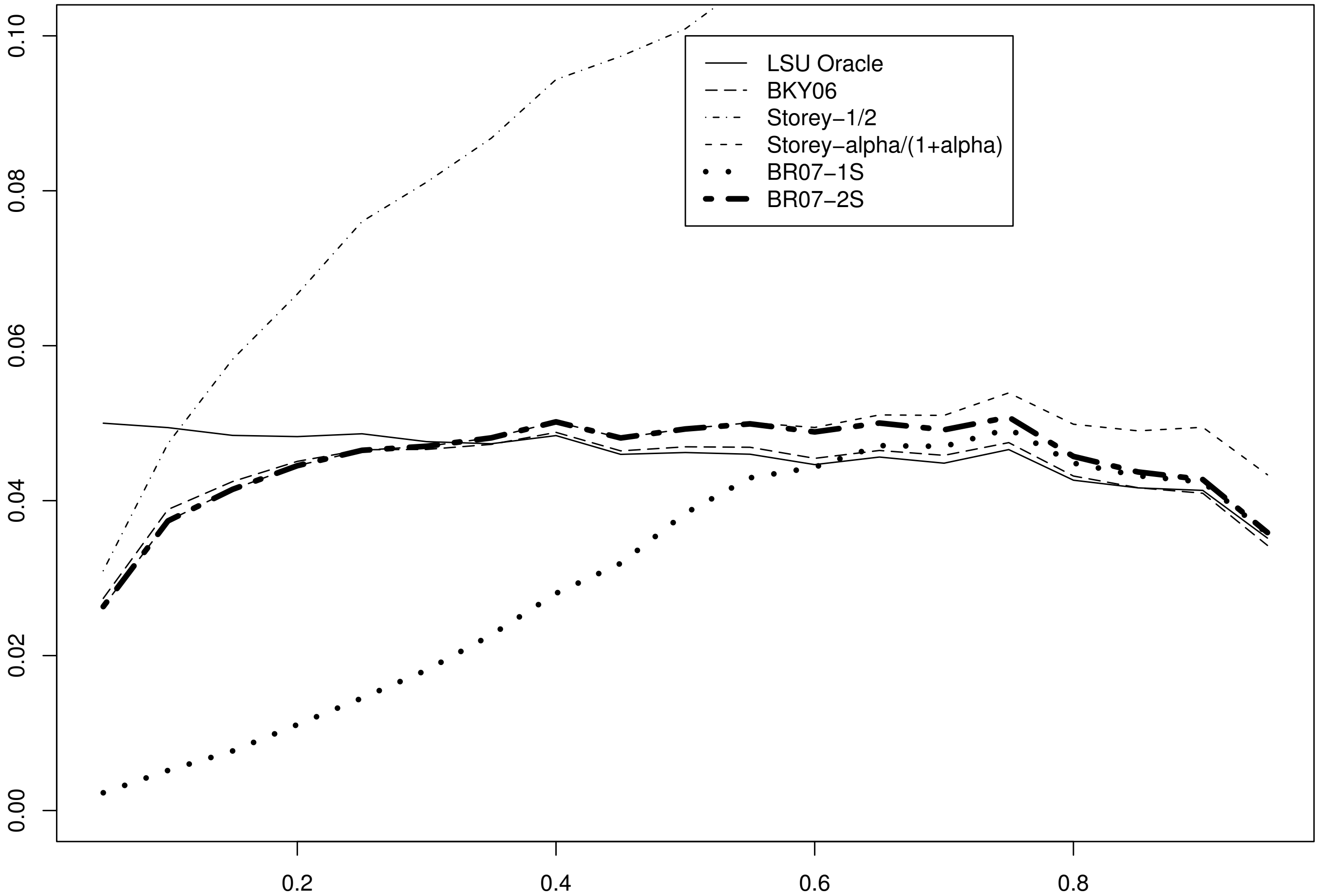

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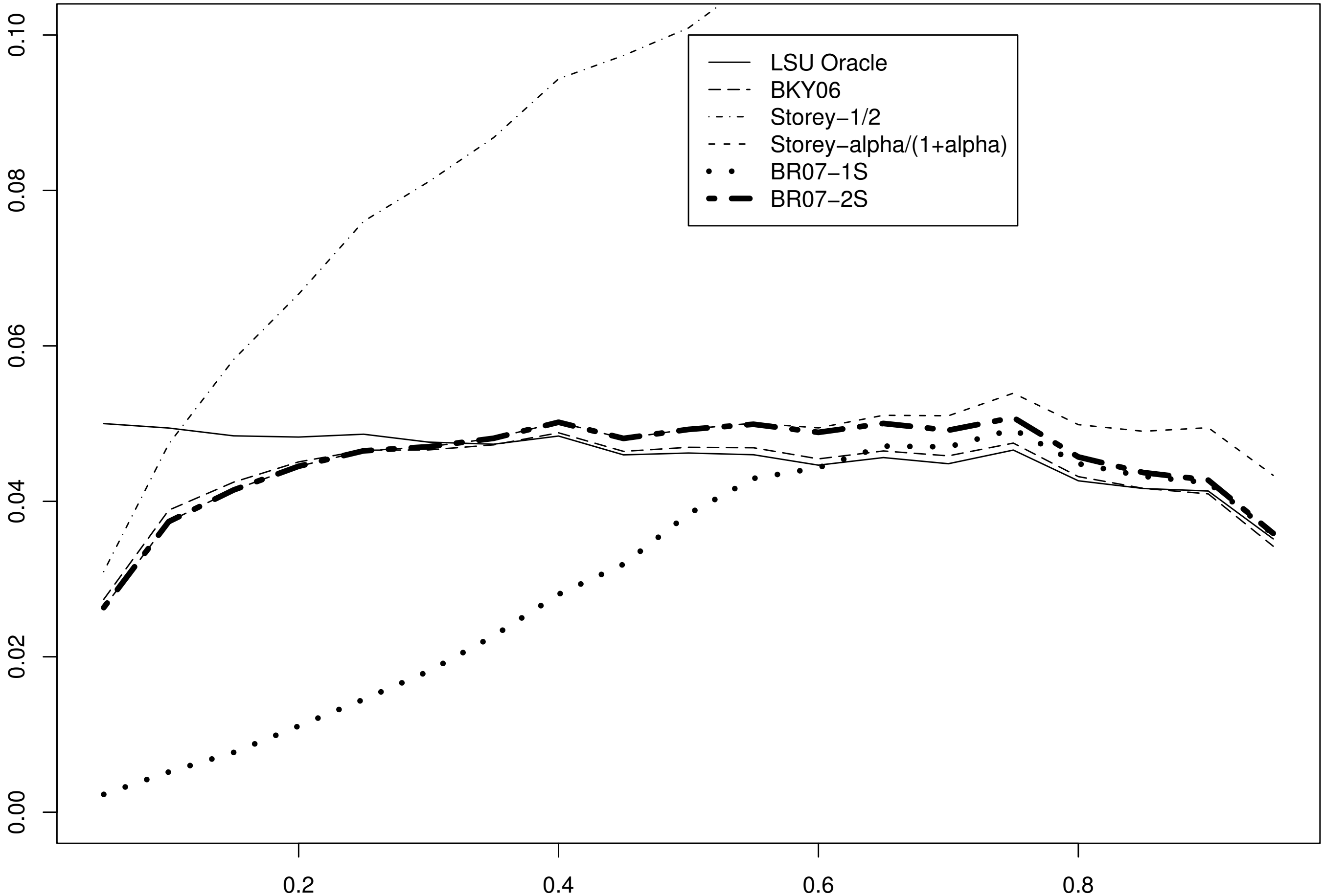

$\Rightarrow$ Storey-$1/2$ is not robust! Storey-$\alpha/(1+\alpha)$ not robust? (max $\simeq 0.054$)

$\Rightarrow$ New procedures seem robust to positive correlations

I I. Simulations

$m = 100$, $m_0 = 5$, $\rho = 0.1$, $2 \leq \text{mean} \leq 5$