The multiple confidence procedure and its applications

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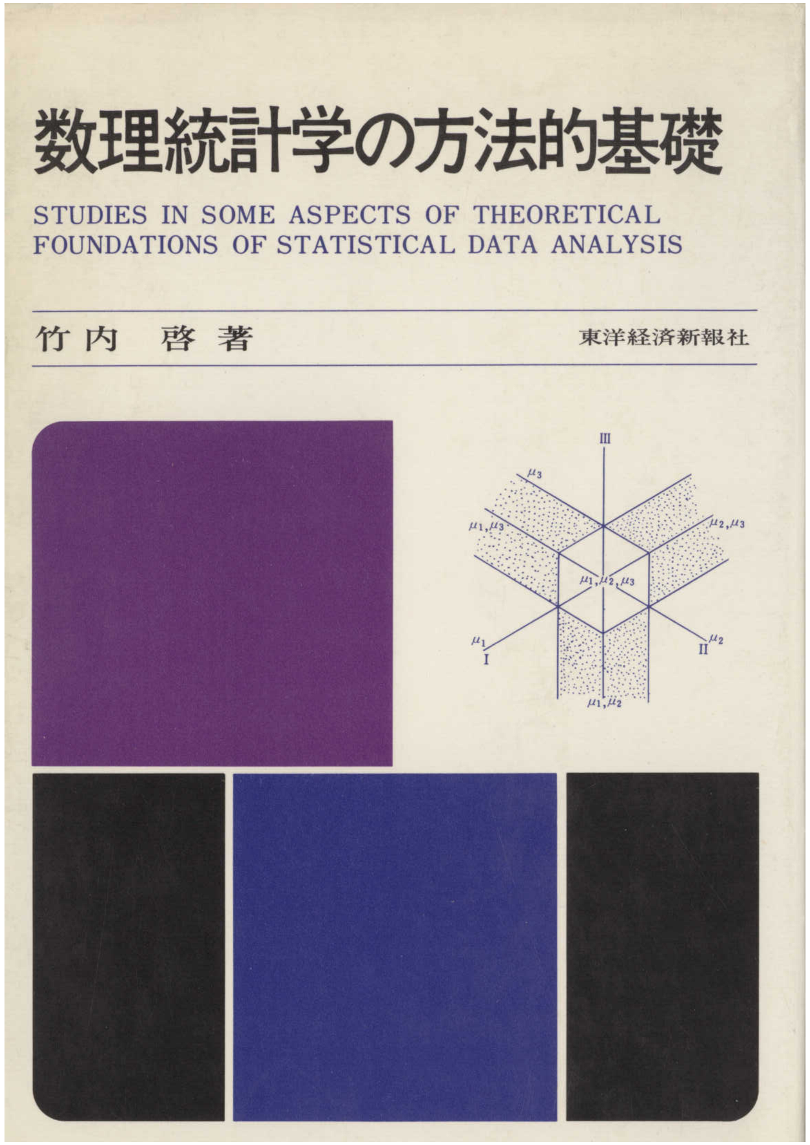
OUTLINE

1 The multiple confidence procedure
by K. Takeuchi (1973)

2 Simple Examples

3 Comparisons between 3 pesticide treatments
Placebo, Control, Test treatments

1 The multiple confidence procedure by K. Takeuchi (1973)



Kei Takeuchi (1973).
***Studies in some aspects of theoretical foundations of statistical data analysis*, Tokyo: Toyo Keizai Shinpo-sha (in Japanese).**

Chapter 8:
Confidence procedures for multiple decision problems

- $\{\mathcal{X}, \Omega, \mathcal{P}\}$

  $X \in \mathcal{X}$: sample space

  $\theta \in \Omega$: parameter space

  $P_\theta \in \mathcal{P}$: probability measures

- family of subsets $\omega \subseteq \Omega$

  $S = \{\omega \mid \omega \subseteq \Omega\}$

Definition 1: multiple decision procedure

mapping $\psi\colon \mathcal{X} \longrightarrow S \quad (\psi(X) \in S)$

- $S = \{\{\theta\}\} \rightarrow$ point estimation
- $S = \{\omega_0,\ \Omega \setminus \omega_0\} \rightarrow$ hypothesis test
- $S = 2^\Omega \rightarrow$ regional estimation

Definition 2: **(multiple) confidence procedure with confidence system $\{1-\alpha_\omega\}$**

$$P_\theta\{\theta \in \psi(X) \mid \theta \in \omega\} \geq 1-\alpha_\omega, \forall\theta \tag{1}$$

- consistent system $\{1-\alpha_\omega\}$
  $\omega' \subseteq \omega \Rightarrow 1-\alpha_{\omega'} \geq 1-\alpha_\omega$
- In many applications, $\alpha_\omega \equiv \alpha$.

Our objective is to construct a multiple confidence procedure with given confidence system $\{1-\alpha_\omega\}$.

Note: If we choose $\psi'(X) \equiv \Omega$, then ψ' is a multiple confidence procedure with confidence system $\{1-\alpha_\omega = 1\}$.

Definition 3: primary element ω_e (minimal element)

There exists no $\omega' \in S,\ \omega' \subsetneq \omega_e$.

$S_e = \{\omega_e\} \subseteq S$: family of all primary elements

Definition 4: S is unrestricted if

$$\bigcup_{\omega_e \in S_e} \omega_e = \Omega \text{ and } \bigcup_{\omega_e \in S'_e \subseteq S_e} \omega_e \in S.$$

Extension $\bar{S}$ of restricted S:

$\bar{S} \supseteq S$, and $\bar{S}$ is unrestricted.

If S is restricted (not unrestricted), we can always construct its extension $\bar{S}$.

Theorem 1: Suppose S is unrestricted.

(1) For any primary element $\omega_e \in S_e$, let $\phi_{\omega_e}(X)$ be a test function for the null hypothesis $H_{\omega_e}: \theta \in \omega_e$ at level α_{ω_e}: $\sup_{\theta\in\omega_e} P_\theta\{\phi_{\omega_e}(X) = 1\} \leq \alpha_{\omega_e}$. Then

$$\psi(X) = \bigcup \{\omega_e \mid \omega_e \in S_e,\ \phi_{\omega_e}(X) = 0\} \qquad (2)$$

is a multiple confidence procedure with confidence system $\{1 - \alpha_\omega\}$, where $\alpha_\omega = \sup_{\omega_e \cap \omega \neq \emptyset} \alpha_{\omega_e}$.

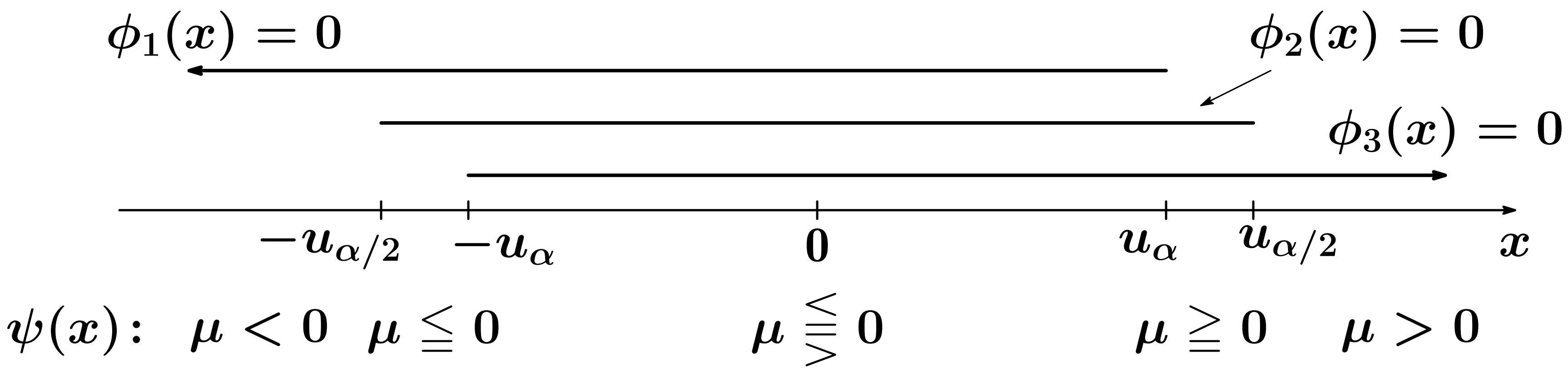
(2) If S consists of only unions of primary elements, any multiple confidence procedure ψ can be expressed in the form of (2).

2 Simple Examples

Example 1 (Takeuchi, 1973):
Suppose $x \sim N(\mu,\ \sigma^2 = 1)$, $\mu \in (-\infty, \infty) = \Omega$, and we are interested in the sign of μ.

primary elements: $S_e = \{\omega_1, \omega_2, \omega_3\}$

$$\begin{cases} \omega_1 : \mu < 0 \longrightarrow \phi_1(x) = 0 \text{ for } x \leq u_\alpha \\ \omega_2 : \mu = 0 \longrightarrow \phi_2(x) = 0 \text{ for } |x| \leq u_{\alpha/2} \\ \omega_3 : \mu > 0 \longrightarrow \phi_3(x) = 0 \text{ for } -u_\alpha \leq x \end{cases}$$



Example 2 (Miwa & Hayter, 1999):
Suppose $x \sim N(\mu, 1)$, $\mu \in (-\infty, \infty) = \Omega$, and we want to construct 2-sided confidence intervals for μ, suspecting $\mu \geq 0$.

primary elements: $\omega_e = \{\mu\},\ S_e = \{\{\mu\}\}$

$$\begin{cases} \mu \leq 0 \longrightarrow \phi_\mu(x) = 0 \text{ for } x - \mu \leq u_\alpha \\ \mu > 0 \longrightarrow \phi_\mu(x) = 0 \text{ for } |x - \mu| \leq u_{\alpha/2} \end{cases}$$

Then we have the following two-sided confidence intervals as a multiple confidence procedure ψ

$$\begin{cases} x > u_{\alpha/2}, & \psi(x) = [x - u_{\alpha/2}, x + u_{\alpha/2}] \\ u_\alpha < x \leq u_{\alpha/2}, & \psi(x) = (0, x + u_{\alpha/2}] \\ -u_{\alpha/2} < x \leq u_\alpha, & \psi(x) = [x - u_\alpha, x + u_{\alpha/2}] \\ x \leq -u_{\alpha/2}, & \psi(x) = [x - u_\alpha, 0] \end{cases}$$

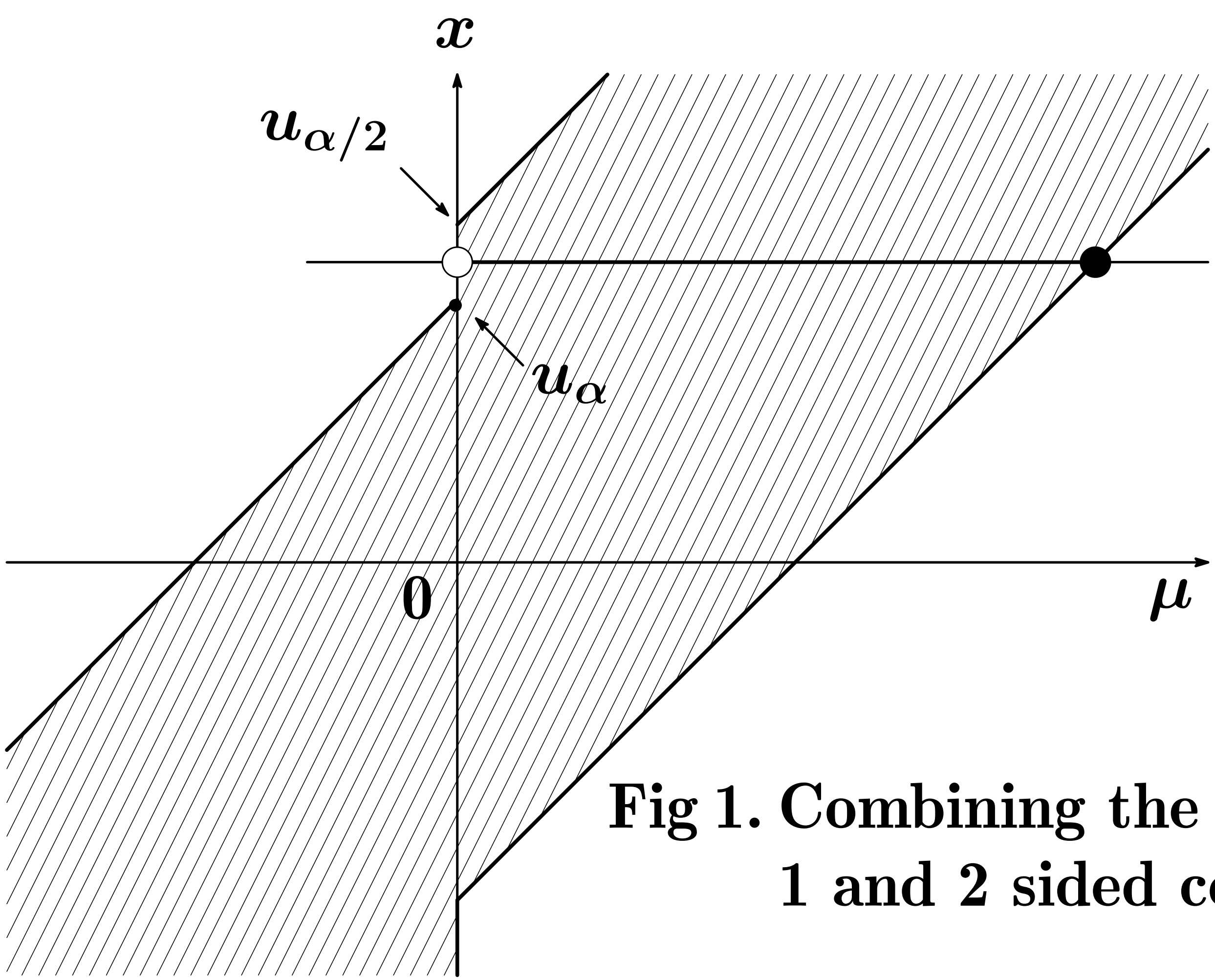


Fig 1. Combining the advantages of 1 and 2 sided confidence intervals

This procedure maintains the same power to detect $\mu > 0$, and provides 2-sided confidence intervals (Miwa & Hayter, 1999).

In the previous construction of ψ

$$\psi(X) = \bigcup \{\omega_e \mid \omega_e \in S_e,\ \phi_{\omega_e}(X) = 0\}$$

we may have some complicated and unintelligible results.

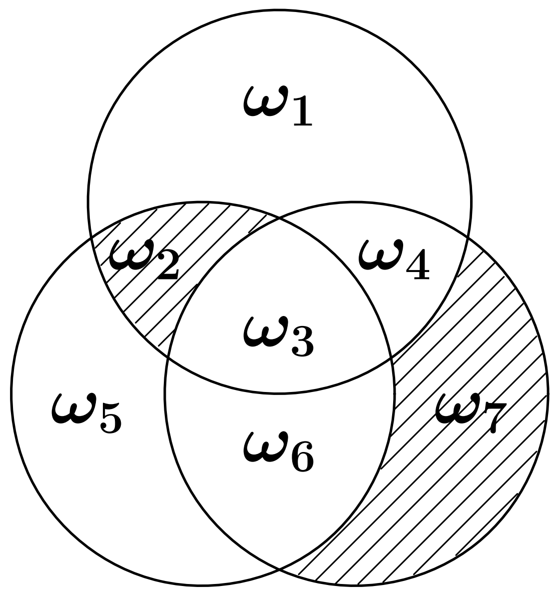


Fig 2. Complicated, unintelligible results

Theorem 2 (Takeuchi, 1973): Suppose $S \ni \Omega$, and let ψ_0 be any multiple confidence procedure for S_0. Then

$$\psi(X) = \omega \supseteq \psi_0(X) \mid \omega \in S$$

is a multiple confidence procedure with

$$\alpha_\omega = \sup_{\omega' \in S_0,\ \omega' \subseteq \omega} \alpha'_\omega.$$

3 Comparisons between 3 pesticide treatments Placebo, Control, Test treatments

- **Three treatment experiments:** $\boldsymbol{\theta} = (\boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \boldsymbol{\mu}_T, \boldsymbol{\sigma}^2)$

$$\begin{cases} \bar{y}_0 \sim N(\mu_0, \sigma^2) & \text{no application (placebo)} \\ \bar{y}_1 \sim N(\mu_1, \sigma^2) & \text{control chemical} \\ \bar{y}_T \sim N(\mu_T, \sigma^2) & \text{new test chemical} \end{cases}$$

Table 1. Possible decisions

decisions	parameter configurations	primary elements
A	$\mu_1 + \Delta_U < \mu_T$	$H_A: \theta \in \omega_A$
B	$\mu_1 - \Delta_L \leq \mu_T \leq \mu_1 + \Delta_U$	$H_B: \theta \in \omega_B$
C	$\mu_0 < \mu_T < \mu_1 - \Delta_L$	$H_C: \theta \in \omega_C$
D	$\mu_T \leq \mu_0$	$H_D: \theta \in \omega_D$

Δ_L, Δ_U are known constants.

- **To apply the multiple confidence procedure, we need to construct a test procedure for interval null hypotheses such as**

$$H_B: -\Delta_L \le \mu_T - \mu_1 \le +\Delta_U.$$

In general, we need a test procedure for

$$x \sim N(\mu, \sigma^2), \quad \hat{\sigma}^2 \sim \sigma^2 \chi^2(\nu)/\nu$$

$$\begin{cases} \text{Null } H_0: & \mu_L \le \mu \le \mu_U \\ \text{Alternative } H_1: & \mu < \mu_L \text{ or } \mu > \mu_U \end{cases}$$

- **Reject H_0 when**

$$\begin{cases} x > \mu_U + \hat{\sigma} \cdot t'(\nu, \delta; \alpha) & \text{or} \\ x < \mu_L - \hat{\sigma} \cdot t'(\nu, \delta; \alpha) \end{cases} \tag{3}$$

- **This test is unbiased, and the critical value $t'(\nu, \delta; \alpha)$ is calculated at $\mu = \mu_U$ (or $\mu = \mu_L$) such that**

$$
\begin{aligned}
&\Pr\{x > \mu_U + \hat{\sigma} \cdot t' \mid \mu = \mu_U\} \\
&+ \Pr\{x < \mu_L - \hat{\sigma} \cdot t' \mid \mu = \mu_U\} \\
&= \Pr\{T(\nu, 0) > t'\} + \Pr\{T(\nu, \delta) < -t'\} \\
&= \alpha,
\end{aligned}
$$

where $T(\nu, \delta)$ is a non-central t variable with non-centrality parameter $\delta = (\mu_U - \mu_L)/\sigma$.

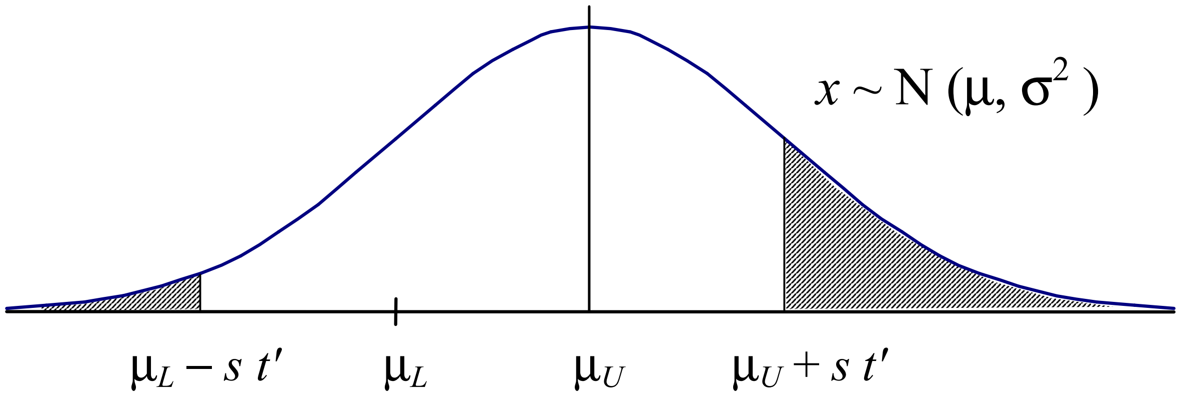


Table 2. Critical values for interval hypotheses

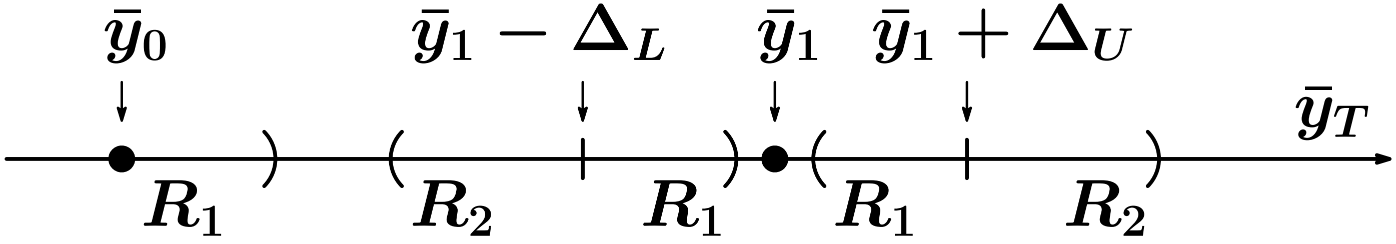
Critical values t'(nu, delta; 0.05) for interval null hypothesis

delta / nu	0.0	0.2	0.4	0.6	0.8	1.0	1.5	2.0	3.0	5.0	Inf
1	12.706	11.235	10.013	9.024	8.248	7.656	6.787	6.451	6.320	6.314	6.314
2	4.303	3.971	3.696	3.476	3.307	3.180	3.003	2.942	2.921	2.920	2.920
3	3.182	2.975	2.805	2.671	2.569	2.494	2.395	2.363	2.354	2.353	2.353
4	2.776	2.610	2.476	2.370	2.291	2.234	2.161	2.138	2.132	2.132	2.132
5	2.571	2.425	2.307	2.215	2.147	2.099	2.038	2.020	2.015	2.015	2.015
6	2.447	2.313	2.205	2.122	2.060	2.017	1.962	1.947	1.943	1.943	1.943
7	2.365	2.238	2.137	2.059	2.001	1.961	1.912	1.898	1.895	1.895	1.895
8	2.306	2.185	2.088	2.014	1.959	1.921	1.875	1.863	1.860	1.860	1.860
9	2.262	2.145	2.052	1.980	1.928	1.892	1.848	1.836	1.833	1.833	1.833
10	2.228	2.114	2.024	1.954	1.903	1.868	1.826	1.815	1.813	1.812	1.812
11	2.201	2.090	2.001	1.933	1.884	1.850	1.809	1.798	1.796	1.796	1.796
12	2.179	2.070	1.982	1.916	1.868	1.834	1.795	1.785	1.782	1.782	1.782
13	2.160	2.053	1.967	1.902	1.854	1.822	1.783	1.773	1.771	1.771	1.771
14	2.145	2.039	1.954	1.890	1.843	1.811	1.773	1.763	1.761	1.761	1.761
15	2.131	2.026	1.943	1.879	1.833	1.802	1.764	1.755	1.753	1.753	1.753
16	2.120	2.016	1.933	1.870	1.825	1.794	1.757	1.748	1.746	1.746	1.746
17	2.110	2.007	1.925	1.862	1.817	1.787	1.751	1.742	1.740	1.740	1.740
18	2.101	1.999	1.917	1.855	1.811	1.780	1.745	1.736	1.734	1.734	1.734
19	2.093	1.991	1.911	1.849	1.805	1.775	1.740	1.731	1.729	1.729	1.729
20	2.086	1.985	1.905	1.844	1.800	1.770	1.735	1.727	1.725	1.725	1.725
24	2.064	1.965	1.886	1.827	1.784	1.755	1.721	1.713	1.711	1.711	1.711
30	2.042	1.945	1.868	1.810	1.768	1.739	1.707	1.699	1.697	1.697	1.697
40	2.021	1.926	1.850	1.793	1.752	1.725	1.693	1.685	1.684	1.684	1.684
60	2.000	1.907	1.833	1.777	1.737	1.710	1.679	1.672	1.671	1.671	1.671
120	1.980	1.888	1.815	1.761	1.722	1.696	1.666	1.659	1.658	1.658	1.658
Inf	1.960	1.870	1.799	1.745	1.707	1.681	1.653	1.646	1.645	1.645	1.645

- **test of each primary element**

$$
\begin{cases}
H_A:\ \mu_1 + \Delta_U < \mu_T & \Rightarrow \text{1-sided test} \\
H_B:\ \mu_1 - \Delta_L \le \mu_T \le \mu_1 + \Delta_U & \Rightarrow \text{interval test} \\
H_C:\ \mu_0 < \mu_T < \mu_1 + \Delta_L & \Rightarrow \text{interval test} \\
 & \quad \text{(1-sided test)} \\
H_D:\ \mu_T \le \mu_0 & \Rightarrow \text{1-sided test}
\end{cases}
$$

$$
R_1 = \sqrt{2/r} \cdot \hat{\sigma} \cdot t(\nu; \alpha),\ R_2 = \sqrt{2/r} \cdot \hat{\sigma} \cdot t'(\nu, \delta; \alpha)
$$



Accepted nulls	H_D, H_C	H_C	H_C, H_B	H_B	H_B, H_A	H_A

Table 3. Resuluts of the multiple confidence procedure

sample mean configurations	accepted nulls	decisions
$\bar{y}_T \leq \bar{y}_0 + R_1$	H_D, H_C	D
$\bar{y}_0 + R_1 < \bar{y}_T < \bar{y}_1 - \Delta_L - R_2$	H_C	C
$\bar{y}_1 - \Delta_L - R_2 \leq \bar{y}_T \leq \bar{y}_1 - \Delta_L + R_1$	H_C, H_B	C
$\bar{y}_1 - \Delta_L + R_1 < \bar{y}_T < \bar{y}_1 + \Delta_U - R_1$	H_B	B
$\bar{y}_1 + \Delta_U - R_1 \leq \bar{y}_T \leq \bar{y}_1 + \Delta_U + R_2$	H_B, H_A	B
$\bar{y}_1 + \Delta_U + R_2 < \bar{y}_T$	H_A	A

Vielen Dank.