Screening Procedures to Identify Robust Product (Process) Designs

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Outline

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1. The Introduction

Goal: Show how the Gupta idea of screening using *subset selection* can be used to determine minimax robust combinations of control variables for products having control & noise variables.

Review

Screening: Balanced One-way Layout

$$Y_{ij} \sim N(\mu_i, \sigma^2),\ 1 \le i \le t,\ 1 \le j \le n$$

where $\mu_1, \ldots, \mu_t$ and σ^2 are unknown.

NT: $\mu_{[1]} \le \cdots \le \mu_{[t]}$

Screening Goal: Given $\alpha, 1/t < 1 - \alpha < 1$, determine a (random) set of treatments, say $\mathcal{G} = \mathcal{G}(\boldsymbol{Y})$, so that

$$P\{[t] \in \mathcal{G}(\boldsymbol{Y}) | \boldsymbol{\mu}, \sigma^2\} \ge 1 - \alpha$$

for *all* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_t)$ and $\sigma^2 > 0$.

Statistical Screening:

$$\overline{Y}_i = \frac{1}{n}\sum_{j=1}^{n} Y_{ij} = i^{th} \text{ sample mean, } 1 \le i \le t$$

NT: $\overline{Y}_{[1]} \le \cdots \le \overline{Y}_{[t]}$

Fact: $\mathcal{G} = \left\{ i : \overline{Y}_i \ge \overline{Y}_{[t]} - hS\sqrt{\frac{2}{n}} \right\}$ is a $100(1-\alpha)\%$ confidence level screening procedure for treatment associated with $\mu_{[t]}$ (with h an upper α critical point from a certain multivariate t-distribution)

$$\overline{Y}_{[t]} - h\,S\sqrt{\frac{2}{n}} \qquad \overline{Y}_{[t]}$$

Features:

a. *Analysis Tool:* works for any sample size n (but width of screening yardstick $\uparrow$ with the measurement error (S) and $\downarrow$ in the sample size n.)

b. *Confidence Interval Flavor*: Confidence level guarantee holds for all $\boldsymbol{\mu}, \boldsymbol{\sigma}^2$

c. Probability of CS depends on $\boldsymbol{\mu}$ and σ^2, i.e.,

$$P\{[t] \in \mathcal{G}(\boldsymbol{Y}) | \boldsymbol{\mu}, \sigma^2\} \ge 1 - \alpha$$

with $\gg$ possible for $\boldsymbol{\mu}$ "favorable" to $\mu_{[t]}$

(discrete distribution confidence interval!)

Our Setup

- *Control (manufacturing, engineering) variables*

$$\boldsymbol{x_c} = (x_1, \ldots, x_g)$$

- *Noise (environmental, field, operating) variables*

$$\boldsymbol{x_e} = (x_1, \ldots, x_f)$$

Consider designing a prosthetic hip to be used in a population of arthritic patients. The performance of the device, Y, is measured by the maximum strain produced at the bone/prosthesis boundary. Distribution Y depends on

Typical $\boldsymbol{x_c}$

- prosthesis geometry
- stiffness (material)

Typical $\boldsymbol{x_e}$

- forces exerted on the prosthesis
- patient bone quality (bone elasticities)
- other environmental variables

Goal: Select a combination of control variables that is robust to the effect of the noise variables

Robust in what sense?

This talk – concerned with applications where it is critical to avoid engineering designs, i.e., $\boldsymbol{x_c}$, for which *there exist levels of the noise variables that can* cause very poor performance.

Example Hip prosthesis designs that perform poorly for segments of patient populations should not be used.

Example 2 Box and Jones (1992) describe a study whose goal is to *improve* the *taste* of a cake mix that consumers bake under conditions that can vary from the directions printed on the cakebox.

- ***Control factors***: amounts of
  - Shortening (S)
  - Flour (F)
  - Egg (E)

  (used in the cake mix)
- ***Noise factors***:
  - baking temperature (T);
  - baking time (Z).

  (are baking directions)

All Control and Noise factors variables treated as qualitative with two levels:

0 = low level

1 = high level

- ***Response*** $= Y_{ijk,lm} =$ taste test rating when $(S, F, E, T, Z) = (ijklm) \in \{0, 1\}^5$ (larger taste test ratings correspond to *better* tasting product)

  ($\mu_{ijk,lm} =$ mean response of $Y_{ijk,lm}$)

Mean Model:

$$\mu_{ijk,lm} = m_0 + S_i + F_j + E_k + T_l + Z_m + (ST)_{il}$$

(T, Z)

$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(S,$ F,E)
$\mu_{000,00}$	$\mu_{000,01}$	$\mu_{000,10}$	$\mu_{000,11}$	(0,0,0)
$\mu_{001,00}$	$\mu_{001,01}$	$\mu_{001,10}$	$\mu_{001,11}$	(0,0,1)
$\mu_{010,00}$	$\mu_{010,01}$	$\mu_{010,10}$	$\mu_{010,11}$	(0,1,0)
$\mu_{011,00}$	$\mu_{011,01}$	$\mu_{011,10}$	$\mu_{011,11}$	(0,1,1)
$\mu_{100,00}$	$\mu_{100,01}$	$\mu_{100,10}$	$\mu_{100,11}$	(1,0,0)
$\mu_{101,00}$	$\mu_{101,01}$	$\mu_{101,10}$	$\mu_{101,11}$	(1,0,1)
$\mu_{101,00}$	$\mu_{101,01}$	$\mu_{101,10}$	$\mu_{101,11}$	(1,1,0)
$\mu_{111,00}$	$\mu_{111,01}$	$\mu_{111,10}$	$\mu_{111,11}$	(1,1,1)

Question: What recipe to use? A maximin approach

$$\xi_{ijk} = \min_{l,\, m} \mu_{ijk,lm}$$

be the worst-case performance of the recipe (i, j, k)

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$$\xi_{ijk} \equiv \min_{l,\, m} \mu_{ijk,lm}$$

be the worst-case performance of the recipe (i, j, k)

Goal: find recipe that ***maximizes*** ξ_{ijk}

A $2^{5\text{-}1}$ Fractional Factorial *Experiment* with defining contrast $I = SFETZ$

(T,Z): $(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	(S, F,E)
–	1.2	1.6	–	(0,0,0)
1.6	–	–	4.4	(0,0,1)
2.2	–	–	6.5	(0,1,0)
–	4.9	6.1	–	(0,1,1)
1.3	–	–	1.7	(1,0,0)
–	2.6	2.4	–	(1,0,1)
–	3.8	3.5	–	(1,1,0)
5.2	–	–	6.0	(1,1,1)

2. The General Formulation

Conduct a balanced full or orthogonal fractional factorial experiment with each factor at levels 0 and 1

- Control factors are indexed by $\boldsymbol{i}$
- Noise factors are indexed by $\boldsymbol{j}$
- $Y_{\boldsymbol{i},\boldsymbol{j}}$ denote the response of interest.

Let

$$\mu_{\boldsymbol{i},\boldsymbol{j}} = \text{mean response of } Y_{\boldsymbol{i},\boldsymbol{j}}$$

when

- $\boldsymbol{x}_c$ is at combination $\boldsymbol{i}$
- $\boldsymbol{x}_e$ is at combination $\boldsymbol{j}$

Mean Model $\mathcal{M}$

Every mean model for $\mu_{\boldsymbol{i},\boldsymbol{j}}$ has 3 types of terms:

- control variables ($\# = p$) that interact with noise variables ($\# = s$)
- control variables that have no interactions with noise variables ($\# = q$)
- noise variables that have no interactions with control variables ($\# = t$)

So

- total # control variables $= p + q$
  with $\boldsymbol{i} = (\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{i}^{\sim}) \in \mathbb{I} = \{0,1\}^{p+q}$
  $\boldsymbol{i}^{\mathcal{I}} \in \mathbb{I}^{\mathcal{I}} = \{0,1\}^{p}$
  $\boldsymbol{i}^{\sim} \in \mathbb{I}^{\sim} = \{0,1\}^{q}$

and

- total # noise variables $= s + t$
  with $\boldsymbol{j} = (\boldsymbol{j}^{\mathcal{I}}, \boldsymbol{j}^{\sim}) \in \mathbb{J} = \{0,1\}^{s+t}$
  $\boldsymbol{j}^{\mathcal{I}} \in \mathbb{J}^{\mathcal{I}} = \{0,1\}^{s}$
  $\boldsymbol{j}^{\sim} \in \mathbb{J}^{\sim} = \{0,1\}^{t}$

Notation	Meaning
$\mathrm{C}_1^{\mathcal{I}}, \ldots, \mathrm{C}_p^{\mathcal{I}}$	Control factors that interact w/ Noise factors
$N_1^{\mathcal{I}}, \ldots, N_s^{\mathcal{I}}$	Noise factors that interact w/ Control factors
$\mathrm{C}_1^{\sim}, \ldots, \mathrm{C}^{\sim}$	Control factors that don't interact w/ Noise Factors
$N_1^{\sim}, \ldots, N_t^{\sim}$	Noise factors that don't interact w/ Control factors

Model $\mathcal{M}$

$$\mu_{\boldsymbol{i},\boldsymbol{j}} = \mu_{cn}(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}) + \mu_c(\boldsymbol{i}^{\sim}) + \mu_n(\boldsymbol{j}^{\sim})$$

where

$$\mu_{cn}(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}) = m_0 + \sum_{Q^{cn}} (\mathrm{C}_{i_1}^{\mathcal{I}}, \ldots, \mathrm{C}_{i_{p*}}^{\mathcal{I}} N_{j_1}^{\mathcal{I}}, \ldots, N_{j_{s*}}^{\mathcal{I}})_{\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}}$$

$$\mu_c(\boldsymbol{i}^{\sim}) = \sum_{Q^c} (\mathrm{C}_{i_1}^{\sim}, \ldots, \mathrm{C}_{i_{q*}}^{\sim})_{\boldsymbol{i}^{\sim}}$$

$$\mu_n(\boldsymbol{j}^{\sim}) = \sum_{Q^n} (N_{i_1}^{\sim}, \ldots, N_{i_{t*}}^{\sim})_{\boldsymbol{j}^{\sim}}$$

where Q^{cn}, Q^c, Q^n denote the sets of main effects and interactions present

Example 2 (Box and Jones – continued)

$$\begin{aligned}\mu_{ijk,lm} &= m_0 + S_i + F_j + E_k + T_l + Z_m + (ST)_{il} \\ &= \{m_0 + S_i + T_l + (ST)_{il}\} \quad [\mu_{cn}(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}})] \\ &\quad + \{F_j + E_k\} \qquad [\mu_c(\boldsymbol{i}^{\sim})] \\ &\quad + \{Z_m\} \qquad [\mu_n(\boldsymbol{j}^{\sim})]\end{aligned}$$

has $p = 1$, $q = 2$, $s = 1$, $t = 1$
(i) (j, k) (l) (m)

Definition: The worst case performance of the engineering design $\boldsymbol{i} \in \mathbb{I}$ is

$$\xi_{\boldsymbol{i}} \equiv \min_{\boldsymbol{j} \in \mathbb{J}} \mu_{\boldsymbol{i},\boldsymbol{j}}$$

Data:

$$Y_{\mathbb{D}} = \mu_{\mathbb{D}} + \epsilon_{\mathbb{D}}$$

where $\mathbb{D}$ is the set of control and noise factor combinations $(\boldsymbol{i},\boldsymbol{j})$ at which observations are taken, $\mu_{\mathbb{D}}$ satisfies the mean model $\boldsymbol{\mathcal{M}}$ for all $(\boldsymbol{i},\boldsymbol{j})$, and $\epsilon_{\mathbb{D}}$ are independent $N(0, \sigma^2)$ measurement errors.

Goal: Choose a set of control factors $\boldsymbol{i}^*$ so that

$$\xi_{\boldsymbol{i}^*} = \max_{\boldsymbol{i}} \xi_{\boldsymbol{i}}$$

based on a full factorial or fractional experiment with data $Y_{\mathbb{D}}$

3. *The Statistical Procedure*

Notation

- $\hat{\mu}_{\boldsymbol{i},\boldsymbol{j}} =$ OLS of $\mu_{\boldsymbol{ij}}$
- $\hat{\xi}_{\boldsymbol{i}} = \min_{\boldsymbol{j}} \hat{\mu}_{\boldsymbol{i},\boldsymbol{j}}$ for $\boldsymbol{i} \in \mathbb{I}$
- ordered $\hat{\xi}_{\boldsymbol{i}}$: $\hat{\xi}_{[1]} \leq \cdots \leq \hat{\xi}_{[|\mathbb{I}|]}$
- $S^2 =$ moment estimator of σ^2

Procedure: Select control factor combination $\boldsymbol{i}$ if and only if

$$\hat{\xi}_{\boldsymbol{i}} \geq \hat{\xi}_{[|\mathbb{I}|]} - h\,S$$

where h is chosen to satisfy

$$P\{\text{Best } \boldsymbol{i} \in \text{selected subset}|\boldsymbol{\mu}, \sigma^2\} \geq 1 - \alpha$$

for all $\boldsymbol{\mu}$ satisfying model $\boldsymbol{\mathcal{M}}$ and $\sigma^2 > 0$, i.e,

$$\inf_{\boldsymbol{\mu} \in \boldsymbol{\mathcal{M}},\, \sigma^2 > 0} P\{CS|\boldsymbol{\mu}, \sigma^2\} \geq 1 - \alpha$$

where CS denotes the event that the best $\boldsymbol{i}$ is in selected subset.

How to Choose h?

We say that $\boldsymbol{\mu}$ satisfies Symmetry Condition $\boldsymbol{\mathcal{S}}$ provided

(1) Whenever an $(\mathrm{C}_{i_1}^{\mathcal{I}}, \ldots, \mathrm{C}_{i_{p*}}^{\mathcal{I}} N_{j_1}^{\mathcal{I}}, \ldots, N_{j_{s*}}^{\mathcal{I}})$ involving p^* control factors and s^* noise factors is present in $\mu_{cn}(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}})$, then every other $p^* + s^*$ factor interaction involving p^*control factors and s^* noise factors must also be in $\mu_{cn}(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}})$

(2) Whenever an interaction $(\mathrm{C}_{i_1}^{\sim}, \ldots, \mathrm{C}_{i_{q*}}^{\sim})$involving q^* control factors is present in $\mu_c(\boldsymbol{i}^{\sim})$, then every other interaction involving q^* control factors must also be present in $\mu_c(\boldsymbol{i}^{\sim})$

(3) Whenever an interaction $(N_{i_1}^{\sim}, \ldots, N_{i_{t*}}^{\sim})$ involvingis t^* noise factors is present in $\mu_n(\boldsymbol{j}^{\sim})$, then every other interaction involving t^* noise factors must also be present in $\mu_n(\boldsymbol{j}^{\sim})$

Theorem Suppose that the model $\mathcal{M}$ satisfies the symmetry condition $\mathcal{S}$, and every main effect and interaction in $\mathcal{M}$ is estimable based on the full factorial or fractional factorial experiment, $\mathbb{D}$. Then if there exists a sequence $\{\boldsymbol{\nu}_k\}_k$ with each $\boldsymbol{\nu}_k$ a $2^p \times 2^s$ matrix satisfying

$$\boldsymbol{\nu}_k(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}) = \mu_{cn}(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}})$$

(of model $\mathcal{M}$) and is such that

$$\lim_{k\to\infty} \boldsymbol{\nu}_k = \begin{pmatrix} 0 & +\infty & \cdots & +\infty \\ \vdots & \vdots & \vdots & \vdots \\ 0 & +\infty & \cdots & +\infty \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then $\boldsymbol{\mu}_k \equiv \boldsymbol{\nu}_k \otimes J_{2^q \times 2^t}$ satisfies

$$[2^{p+q} \times 2^{s+t} = 2^p \times 2^s \otimes 2^q \times 2^t]$$

$$\lim_{k\to\infty} P_{\boldsymbol{\mu}_k}\{ \overset{\wedge}{\xi}_{\mathbf{1}_{p+q}} \geq \overset{\wedge}{\xi}_{[|\mathbb{I}|]} - hS|\boldsymbol{\mu}_k, 1\} = \inf_{\boldsymbol{\mu},\boldsymbol{\sigma}^2} P\{CS|\boldsymbol{\mu}, \sigma^2\}$$

where $\mathbf{1}_m^{\mathrm{T}} = (1, \ldots, 1)$ denotes the unit vector of length m.

Example (Box and Jones – continued) Recall
$\mu_{cn}(i,l) = m_0 + S_i + T_l + (ST)_{il}$ where $i, l \in \{0,1\}$
is a four parameter model and hence every 2×2
$\boldsymbol{\nu}$ matrix satisfies $\mu_{cn}(i,l)$. We can take

$$\boldsymbol{\nu}_k = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & +\infty \\ 0 & 0 \end{pmatrix}$$ so the limiting $\boldsymbol{\mu}_k$ is

$$\begin{pmatrix} 0 & +\infty \\ 0 & 0 \end{pmatrix} \otimes J_{2^2 \times 2^1} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} +\infty & +\infty \\ +\infty & +\infty \\ +\infty & +\infty \\ +\infty & +\infty \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

(for large k, $\boldsymbol{\nu}_k \otimes J_{2^q \times 2^t}$ is "LFC")

Notice: $\xi_{ijk} = 0$ for all $(i,j,k) \in \{0,1\}^3$

$\hat{\xi}_{0jk}$ involves 2, not 4, means $\hat{\mu}_{\boldsymbol{i,j}}$

Some Intuition: Let $(\boldsymbol{i})$ denote the best treatment combination.

$$\inf_{\boldsymbol{\mu}} P\left\{ \hat{\xi}_{(\boldsymbol{i})} \geq \max_{\boldsymbol{i}} \hat{\xi}_{\boldsymbol{i}} - hS | \mu, \sigma^2 \right\}$$

is attained (in the limit) when

- $\hat{\xi}_{(\boldsymbol{i})}$ is stochastically as *small* as possible

and

- the $\hat{\xi}_{\boldsymbol{i}}$, $\boldsymbol{i} \neq (\boldsymbol{i})$ are stochastically *large* as possible.

But

$$\hat{\xi}_{\boldsymbol{i}} = \min_{\boldsymbol{j}} \hat{\mu}_{\boldsymbol{i},\boldsymbol{j}}$$

Note

a. $\hat{\xi}_{\boldsymbol{i}}$ is stochastically large if it involves as *few* $\hat{\mu}_{\boldsymbol{i},\boldsymbol{j}}$ as possible. This is accomplished by setting as many $\hat{\mu}_{\boldsymbol{i},\boldsymbol{j}}$ as possible equal to $+\infty$, consistent with the mean $\boldsymbol{\mu}$ satisfying the model $\boldsymbol{\mathcal{M}}$.

b. $\hat{\xi}_{(\boldsymbol{i})}$ is stochastically as *small* as possible if the $\hat{\mu}_{\boldsymbol{i},\boldsymbol{j}}$ have common mean equal to the $\max_{\boldsymbol{i}} \xi_{\boldsymbol{i}}$

c. When the conditions of the Theorem are satisified.

$$P_{\boldsymbol{\mu}_{\mathbf{LFC}}}\{CS\}=P_{\boldsymbol{\mu}_{\mathbf{0}}}\left\{\begin{array}{l}
\min\{\hat{\mu}_{11100},\hat{\mu}_{11110}\} \ge \hat{\mu}_{00000} - hS \\
\min\{\hat{\mu}_{11100},\hat{\mu}_{11110}\} \ge \hat{\mu}_{00100} - hS \\
\min\{\hat{\mu}_{11100},\hat{\mu}_{11110}\} \ge \hat{\mu}_{01000} - hS \\
\min\{\hat{\mu}_{11100},\hat{\mu}_{11110}\} \ge \hat{\mu}_{01100} - hS \\
\min\{\hat{\mu}_{11100},\hat{\mu}_{11110}\} \ge \min\left\{\hat{\mu}_{10000},\hat{\mu}_{10010}\right\} - hS \\
\min\{\hat{\mu}_{11100},\hat{\mu}_{11110}\} \ge \min\left\{\hat{\mu}_{10100},\hat{\mu}_{10110}\right\} - hS \\
\min\{\hat{\mu}_{11100},\hat{\mu}_{11110}\} \ge \min\left\{\hat{\mu}_{11000},\hat{\mu}_{11010}\right\} - hS \\
\min\{\hat{\mu}_{11100},\hat{\mu}_{11110}\} \ge \min\left\{\hat{\mu}_{11000},\hat{\mu}_{11110}\right\} - hS
\end{array}\right\}$$

where the Y_{ijklm} have mean $\boldsymbol{\mu_0}$ = vector of ***zero*** means, ***unit*** variances. This probability depends only the means

$$\begin{pmatrix}
\hat{\mu}_{00000} & - & - & - \\
\hat{\mu}_{00100} & - & - & - \\
\hat{\mu}_{01000} & - & - & - \\
\hat{\mu}_{01100} & - & - & - \\
\hat{\mu}_{10000} & - & \hat{\mu}_{10010} & - \\
\hat{\mu}_{10100} & - & \hat{\mu}_{10110} & - \\
\hat{\mu}_{11000} & - & \hat{\mu}_{11010} & - \\
\hat{\mu}_{11100} & - & \hat{\mu}_{11110} & -
\end{pmatrix}$$

Example Six factor experiment with variables

Index	***Notation***	***Type***
i	$\mathrm{C}_1^{\mathcal{I}}$	Interacting Control
j, k	$\mathrm{C}_1^{\sim}, \mathrm{C}_2^{\sim}$	Non-Interacting Control
l, m	$N_1^{\mathcal{I}}, N_2^{\mathcal{I}}$	Interacting Noise
n	$N_1^{\sim}$	Non-Interacting Noise

$$\mu_{ijk,lmn} = \mu_{cn}(i, l\, m) + \mu_c(j\, k) + \mu_n(n)$$

where

$$\mu_{cn}(i, l\, m) = m_0 + (\mathrm{C}_1^{\mathcal{I}})_i + (N_1^{\mathcal{I}})_l + (N_2^{\mathcal{I}})_m + (\mathrm{C}_1^{\mathcal{I}} N_1^{\mathcal{I}})_{il} + (\mathrm{C}_1^{\mathcal{I}} N_2^{\mathcal{I}})_{im}$$

satisfies symmetry condition $\boldsymbol{\mathcal{S}}$ and

$$\boldsymbol{\nu_k} = \begin{pmatrix} 0 & k & k & 2k \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ satisfies the model}$$

$$\rightarrow \begin{pmatrix} 0 & +\infty & +\infty & +\infty \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so

$$\begin{pmatrix} 0 & +\infty & +\infty & +\infty \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes J_{2^2 \times 2^1} \text{ is the LFC}$$

4. The Caveats

a. Determining the existence of ν_k :

$$\boldsymbol{\nu}_k \to \begin{pmatrix} 0 & +\infty & \cdots & +\infty \\ \vdots & \vdots & \vdots & \vdots \\ 0 & +\infty & \cdots & +\infty \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

can be done by solving the linear programming problem

$$\max w$$

$$\boldsymbol{s.t.}$$

$$\nu(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}) = 0 \text{ for all } (\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}) \in \mathcal{Z}$$

$$w \leq \nu(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}) \text{ for all } (\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}) \notin \mathcal{Z}$$

$$\sum_{\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}} (\mathrm{C}^{\mathcal{I}}_{i_1}, \ldots, \mathrm{C}^{\mathcal{I}}_{i_{p*}} N^{\mathcal{I}}_{j_1}, \ldots, N^{\mathcal{I}}_{j_{s*}})_{\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}} = 0 \text{ for all}$$

$$(i_1, \ldots, i_{p*}, j_1, \ldots, j_{q*})$$

where $\mathcal{Z}$ are row-col combinations that must be in the limit, and $\nu(\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}})$ is shorthand notation for

$$m_0 + \sum_{Q^{cn}} (\mathrm{C}^{\mathcal{I}}_{i_1}, \ldots, \mathrm{C}^{\mathcal{I}}_{i_{p*}} N^{\mathcal{I}}_{j_1}, \ldots, N^{\mathcal{I}}_{j_{s*}})_{\boldsymbol{i}^{\mathcal{I}}, \boldsymbol{j}^{\mathcal{I}}}$$

b. What if there is no $\boldsymbol{\nu}_k$ with $\begin{pmatrix} 0 & +\infty & \cdots & +\infty \\ \vdots & \vdots & \vdots & \vdots \\ 0 & +\infty & \cdots & +\infty \\ 0 & 0 & 0 & 0 \end{pmatrix}$

as limit? We give a lower bound which has the form of a minimum of a finite set of probabilities computed with zero mean and unit variance, of which the Box & Jones expression is a prototype.

***c. Computing h*:** our recommendation $-$ simulation, eg, Box & Jones example requires the $100 \times (1-\alpha)\%$ quantile of the distribution of

$$T \equiv \frac{\sqrt{9}\left(\max\left\{\begin{array}{llll} \hat{\mu}_{00000}, & \hat{\mu}_{00100}, & \hat{\mu}_{02000}, & \hat{\mu}_{01100}, \\ \min\left\{\hat{\mu}_{10000},\hat{\mu}_{10010}\right\}, & \min\left\{\hat{\mu}_{10100},\hat{\mu}_{10110}\right\}, & & \\ \min\left\{\hat{\mu}_{11000},\hat{\mu}_{11010}\right\}, & \min\left\{\hat{\mu}_{11100},\hat{\mu}_{11110}\right\} & & \end{array}\right\} - \min\left\{\hat{\mu}_{11100},\hat{\mu}_{11110}\right\}\right)}{\sqrt{V}}$$

where $V \sim \chi^2_9$ and is independent of $\boldsymbol{Y}$

d. Bounded Means: Suppose that $L \leq \mu_{\boldsymbol{i},\boldsymbol{j}} \leq U$ for all means $\mu_{\boldsymbol{i},\boldsymbol{j}}$. Then the LFC in Theorem is given by

$$\begin{pmatrix} 0 & U-L & \cdots & U-L \\ \vdots & \vdots & \vdots & \vdots \\ 0 & U-L & \cdots & U-L \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes J_{|\mathcal{I}^{\sim}|\times|\mathcal{J}^{\sim}|}$$

5. Conclusions

a. These results allow a wide variety of QC experiments to be analysed using subset selection methodology to find the "best" product or process

b. ***Weighted Mean Robustness***

Assume that there is *known* a distribution over the noise variables that characterizes some field use (production setting), i.e., there are known non-negative weights w_{j} with $\sum_{j \in \mathcal{J}} w_{j} = 1$, so that

$$\mu_i^w = \sum_{j \in \mathcal{J}} \mu_{i,j} \, w_j$$

is the mean response over multiple uses of the product design i in the field.

Goal: Choose the control variable combination i^* for which

$$\mu_{i*}^w = \max_q \mu_q^w$$

The solution to the problem of finding a mean-robust control variable combination is *simpler* than that of finding a maximin-robust control variable combination.